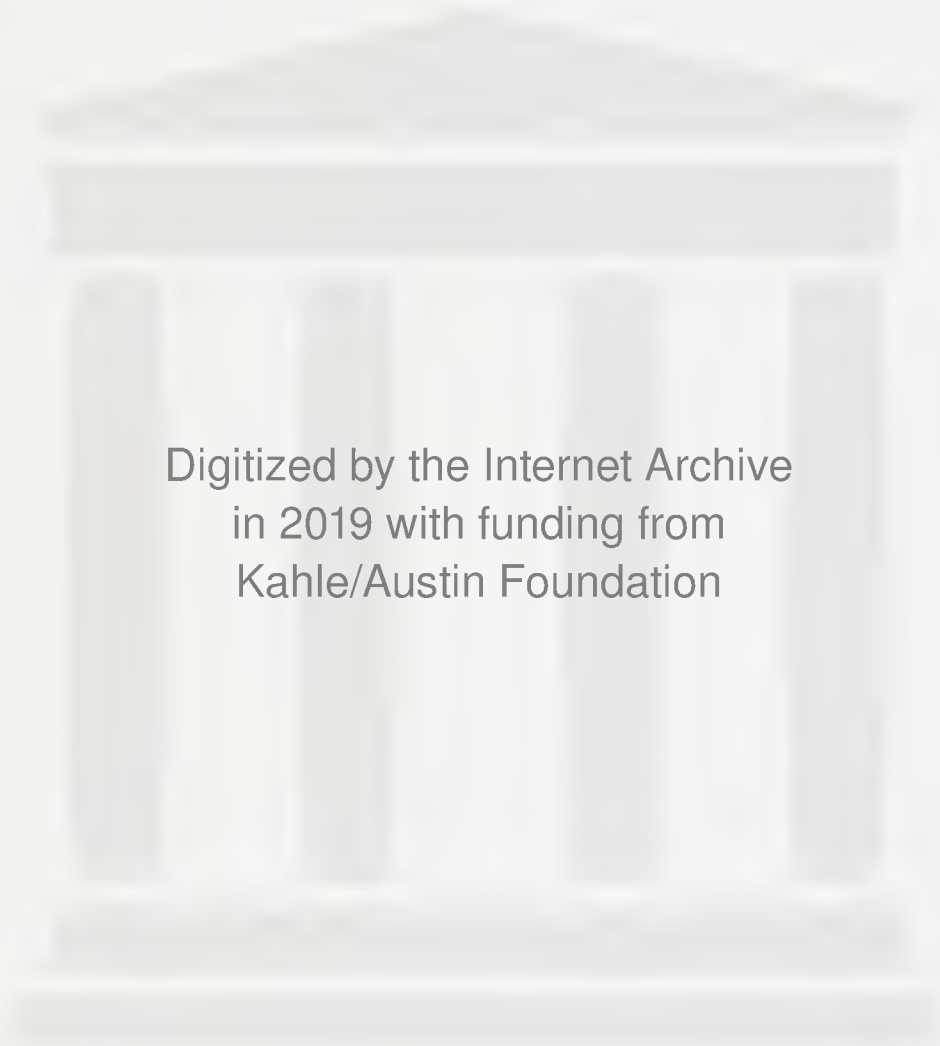

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Space - Time - Matter

Modern Kaluza-Klein Theory

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PREFACE

Einstein endorsed the view of Kaluza, that gravity could be combined with electromagnetism if the dimensionality of the world is extended from 4 to 5. Klein applied this idea to quantum theory, laying a basis for the various modern versions of string theory. Recently, work by a group of researchers has resulted in a coherent formulation of 5D relativity, in which matter in 4D is induced by geometry in 5D. This theory is based on an unrestricted group of 5D coordinate transformations that leads to new solutions and agreement with the classical tests of relativity. This book collects together the main technical results on 5D relativity, and shows how far we can realize Einstein's vision of physics as geometry.

Space, time and matter are physical concepts, with a long but somewhat subjective history. Tensor calculus and differential geometry are highly developed mathematical formalisms. Any theory which joins physics and algebra is perforce open to discussions about interpretation, and the one presented in this book leads to new issues concerning the nature of matter. The present theory should not strictly speaking be called Kaluza-Klein: KK theory relies on conditions of cylindricity and compactification which are now removed. The theory should also, while close to it in some ways, not be confused with general relativity: GR theory has an explicit energy-momentum tensor for matter while now there is none. What we call matter in 4D spacetime is the manifestation of the fifth dimension, hence the phrase induced-matter theory sometimes used in the literature. However, there is nothing sacrosanct about 5D. The field equations take the same form in ND, and N is to be chosen with a view to physics. Thus, superstrings (10D) and supergravity (11D) are valid constructs. However, practical physical applications are expected to be forthcoming only if there is physical understanding of the nature of the extra dimensions and the extra coordinates. In this regard, space-time-matter theory is

uniquely fortunate. This because (unrestricted) 5D Riemannian geometry turns out to be just algebraically rich enough to unify gravity and electromagnetism with their sources of mass and charge. In other words, it is a Machian theory of mechanics.

There is now a large and rapidly growing literature on this theory, and the author is aware that what follows is more like a textbook on basics than a review of recent discoveries. It should also be stated that much of what follows is the result of a group effort over time. Thus credit is due especially to H. Liu, B. Mashhoon and J. Ponce de Leon for their solid theoretical work; to C.W.F. Everitt who sagely kept us in contact with experiment; and to A. Billyard, D. Kalligas, J.M. Overduin and W. Sajko, who as graduate students cheerfully tackled problems that would have made their older colleagues blink. Thanks also go to S. Chatterjee, A. Coley, T. Fukui and R. Tavakol for valuable contributions. However, the responsibility for any errors or omissions rests with the author.

The material in this book is diverse. It is largely concerned with higher-dimensional gravity, touches particle physics, and looks for application to astrophysics and cosmology. Depending on their speciality, some workers may not wish to read this book from cover to cover. Therefore the material has been arranged in approximately self-contained chapters, with a bibliography at the end of each. The material does, of course, owe its foundation to Einstein. However, it will be apparent to many readers that it also owes much to the ideas of his contemporary, Eddington.

Paul S. Wesson

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1. CONCEPTS AND THEORIES OF PHYSICS

"Physics should be beautiful" (Sir Fred Hoyle, Venice, 1974)

1.1 Introduction

Physics is a logical activity, which unlike some other intellectual pursuits frowns on radical departures, progressing by the introduction of elegant ideas which give a better basis for what we already know while leading to new results. However, this inevitably means that the subject at a fundamental level is in a constant state of reinterpretation. Also, it is often not easy to see how old concepts fit into a new framework. A prime example is the concept of mass, which has traditionally been regarded as the source of the gravitational field. Historically, a source and its field have been viewed as separate things. But as recognized by a number of workers through time, this distinction is artificial and leads to significant technical problems. Our most successful theory of gravity is general relativity, which traditionally has been formulated in terms of a set of field equations whose left-hand side is geometrical (the Einstein tensor) and whose right-hand side is material (the energy-momentum tensor). However, Einstein himself realized soon after the formulation of general relativity that this split has drawbacks, and for many years looked for a way to transpose the "base-wood" of the right-hand side of his equations into the "marble" of the left-hand side. Building on ideas of Kaluza and Klein, it has recently become feasible to realize Einstein's dream, and the present volume is mainly a collection of technical results, which shows how this can be done. The basic idea is to unify the source and its field using the rich algebra of higher-dimensional Riemannian geometry. In other words: space, time and matter become parts of geometry.

This is an idea many workers would espouse, but to be something more than an academic jaunt we have to recall the two conditions noted above. Namely, we have to recover what we

already know (with an unavoidable need for reinterpretation); and we have to derive something new with at least a prospect of testability. The present chapter is concerned with the first of these, and the succeeding chapters mainly with the second. Thus the present chapter is primarily a review of gravitation and particle physics as we presently understand these subjects. Since this is mainly known material, these accounts will be kept brief, and indeed those readers who are familiar with these subjects may wish to boost through them. However, there is a theme in the present chapter, which transcends the division of physics into theories of macroscopic and microscopic scope. This is the nature and origin of the so-called fundamental constants. These are commonly taken as indicators of what kind of theory is under consideration (e.g., Newton's constant is commonly regarded as typical of classical theory and Planck's constant as typical of quantum theory). But at least one fundamental constant, the speed of light, runs through all modern physical theories; and we cannot expect to reach a meaningful unification of the latter without a proper understanding of where the fundamental constants originate. In fact, the chapters after this one make use of a view that it is necessary to establish and may be unfamiliar to some workers: the fundamental constants are not really fundamental, their main purpose being to enable us to dimensionally transpose certain material quantities so that we can write down consistent laws of physics.

1.2 Fundamental Constants

A lot has been written on these, and there is a large literature on unsuccessful searches for their possible variations in time and space. We will be mainly concerned with their origin and status, on which several reviews are available. Notably there are the books by Wesson (1978), Petley (1985) and Barrow and Tipler (1986); the conference proceedings edited by McCrea and

Rees (1983); and the articles by Barrow (1981) and Wesson (1992). We will presume a working physicist's knowledge of the constants concerned, and the present section is to provide a basis for the discussions of physical theory which follow.

The so-called fundamental constants are widely regarded as a kind of distillation of physics. Their dimensions are related to the forms of physical laws, whose structure can in many cases be recovered from the constants by dimensional analysis. Their sizes for some choice of units allow the physical laws to be evaluated and compared to observation. Despite their perceived fundamental nature, however, there is no theory of the constants as such. For example, there is no generally accepted formalism that tells us how the constants originate, how they relate to one another, or how many of them are necessary to describe physics. This lack of background seems odd for parameters that are widely regarded as basic.

The constants we will be primarily concerned with are those that figure in gravity and particle physics. It is convenient to collect the main ones here, along with their dimensions and approximate sizes in c.g.s. units:

Speed of light	c	$L\ T^{-1}$	3.0×10^{10}
Gravitational constant	G	$M^{-1}L^3\ T^{-2}$	6.7×10^{-8}
Planck's constant	h	$M\ L^2\ T^{-1}$	6.6×10^{-27}
Electron charge (modulus)	e	$M^{1/2}L^{3/2}\ T^{-1}$	4.8×10^{-10}.

Here e is measured in electrostatic or Gaussian units. We will use e.s.u. in the bulk of what follows, though S.I. will be found useful in places. The two systems of units are of course related by $4\pi\varepsilon_0$, where the permittivity of free space is $\varepsilon_0 = 8.9 \times 10^{-12} C^2 m^{-3} s^2\ kg^{-1}$. In S.I.

$e = 1.6 \times 10^{-19}$C (Coulombs: see Jackson 1975, pp. 29, 817; and Griffiths 1987, p. 9). The permeability of free space μ_0 is not an independent constant because $c^2 \equiv 1/\varepsilon_0\mu_0$. The above table suggests that we need to understand 3 overlapping things: constants, dimensions and units.

One common view of the constants is that they define asymptotic states. Thus c is the maximum velocity of a massive particle moving in flat spacetime; G defines the limiting potential for a mass that does not form a black hole in curved spacetime; ε_0 is the empty-space or vacuum limit of the dielectric constant; and h defines a minimum amount of energy (alternatively $\hbar \equiv h/2\pi$ defines a minimum amount of angular momentum). This view is acceptable, but somewhat begs the question of the constants' origin.

Another view is that the constants are necessary inventions. Thus if a photon moves away from an origin and attains distance r in time t, it is necessary to write r = ct as a way of reconciling the different natures of space and time. Or, if a test particle of mass m_1 moves under the gravitational attraction of another mass m_2 and its acceleration is d^2r/dt^2 at separation r, it is observed that m_1d^2r/dt^2 is proportional to m_1m_2/r^2, and to get an equation out of this it is necessary to write $d^2r/dt^2 = Gm_2/r^2$ as a way of reconciling the different natures of mass, space and time. A similar argument applies to the motion of charged bodies and ε_0. In quantum theory, the energy E of a photon is directly related to its frequency ν, so we necessarily have to write $E = h\nu$. The point is, that given a law of physics which relates quantities of different dimensional types, constants with the dimensions $c = LT^{-1}$, $G = M^{-1}L^3T^{-2}$, $\varepsilon_0 = Q^2M^{-1}L^{-3}T^2$ and $h = ML^2T^{-1}$ are obligatory.

This view of the constants is logical, but disturbing to many because it means they are not really fundamental and in fact largely subjective in origin. However, it automatically answers the question raised in the early days of dimensional analysis as to why the equations of physics

are dimensionally homogeneous (e.g. Bridgman 1922). It also explains why subsequent attempts to formalize the constants using approaches such as group theory have led to nothing new physically (e.g. Bunge 1971). There have also been notable adherents of the view that the fundamental constants are not what they appear to be. Eddington (1929, 1935, 1939) put forward the opinion that while an external world exists, our laws are subjective in the sense that they are constructed to match our own physical and mental modes of perception. Though he was severely criticized for this opinion by physicists and philosophers alike, recent advances in particle physics and relativity make it more palatable now than before. Jeffreys (1973, pp. 87-94, 97) did not see eye to eye with Eddington about the sizes of the fundamental constants, but did regard some of them as disposable. In particular, he pointed out that in electrodynamics c merely measures the ratio of electrostatic and electromagnetic units for charge. Hoyle and Narlikar (1974, pp. 97, 98) argued that the c^2 in the common relativistic expression $(c^2t^2 - x^2 - y^2 - z^2)$ should not be there, because "there is no more logical reason for using a different time unit than there would be for measuring x, y, z in different units". They stated that the velocity of light is unity, and its size in other units is equivalent to the definition 1 s = 299 792 500 m, where the latter number is manmade. McCrea (1986, p. 150) promulgated an opinion that is exactly in line with what was discussed above, notably in regard to c, h and G, which he regarded as "conversion constants and nothing more". These comments show that there is a case that can be made for removing the fundamental constants from physics.

Absorbing constants in the equations of physics has become commonplace in recent years, particularly in relativity where the algebra is usually so heavy that it is undesirable to encumber it with unnecessary symbols. Formally, the rules for carrying this out in a consistent fashion are well known (see e.g. Desloge 1984). Notably, if there are N constants with N bases,

and the determinant of the exponents of the constants' dimensions is nonzero so they are independent, then their magnitudes can be set to unity. For the constants c, G, ε_0, h with bases M, L, T, Q it is obvious that ε_0 and Q can be removed this way. (Setting $\varepsilon_0 = 1$ gives Heaviside-Lorentz units, which are not the same as setting $4\pi\varepsilon_0 = 1$ for Gaussian units, but the principle is clearly the same: see Griffiths, 1987, p. 9.) The determinant of the remaining dimensional combinations $M^0L^1T^{-1}$, $M^{-1}L^3T^{-2}$, $M^1L^2T^{-1}$ is finite, so the other constants c, G, h can be set to unity. Conceptually, the absorbing of constants in this way prompts 3 comments. (a) There is an overlap and ambiguity between the idea of a base dimension and the idea of a unit. All of mechanics can be expressed with dimensional bases M, L, T; and we have argued above that these originate because of our perceptions of mass, length and time as being different things. We could replace one or more of these by another base (e.g. in engineering force is sometimes used as a base), but there will still be 3. If we extend mechanics to include electrodynamics, we need to add a new base Q. But the principle is clear, namely that the base dimensions reflect the nature and extent of physical theory. In contrast, the idea of a unit is less conceptual but more practical. We will discuss units in more detail below, but for now we point up the distinction by noting that a constant can have different sizes depending on the choice of units while retaining the same dimensions. (b) The process of absorbing constants cannot be carried arbitrarily far. For example, we cannot set e = 1, $\hbar = 1$ and c = 1 because it makes the electrodynamic fine-structure constant $\alpha \equiv e^2/\hbar c$ equal to 1, whereas in the real world it is observed to be approximately 1/137. This value actually has to do with the peculiar status of e compared to the other constants (see below), but the caution is well taken. (c) Constants mutate with time. For example, the local acceleration of gravity g was apparently at one time viewed as a 'fundamental' constant, because it is very nearly the same at all places on the Earth's surface.

But today we know that $g = GM_E / r_E^2$ in terms of the mass and radius of the Earth, thus redefining g in more basic terms. Another example is that the gravitational coupling constant in general relativity is not really G but the combination $8\pi G / c^4$ (Section 1.3), and more examples are forthcoming from particle physics (Section 1.4). The point of this and the preceding comments is that where the fundamental constants are concerned, formalism is inferior to understanding.

To gain more insight, let us discuss in greater detail the relation between base dimensions and units, concentrating on the latter. There are 7 base dimensions in widespread use (Petley 1985, pp. 26-29). Of these 3 are the familiar M, L, T of mechanics. Then electric current is used in place of Q. And the other 3 are temperature, luminous intensity and amount of substance (mole). As noted above, we can swap dimensional bases if we wish as a matter of convenience, but the status of physics fixes their number. By contrast, choices of units are infinite in number. At present there is a propensity to use the S.I. system (Smith 1983). While not enamoured by workers in astrophysics and certain other disciplines because of the awkwardness of the ensuing numbers, it is in widespread use for laboratory-based physics. The latter requires well-defined and reproducible standards, and it is relevant to review here the status of our basic units of time, length and mass.

The second in S.I. units is defined as 9 192 631 770 periods of a microwave oscillator running under well-defined conditions and tuned to maximize the transition rate between two hyperfine levels in the ground state of atoms of ^{133}Cs moving without collisions in a near vacuum. This is a fairly sophisticated definition, which is used because the caesium clock has a long-term stability of 1 part in 10^{14} and an accuracy of reproducibility of 1 part in 10^{13}. These specifications are better than those of any other apparatus, though in principle a water clock

would serve the same purpose. So much for a unit of time. The metre was originally defined as the distance between two scratch marks on a bar of metal kept in Paris. But it was redefined in 1960 to be 1 650 763.73 wavelengths of one of the orange-red lines in the spectrum of a ^{83}Kr lamp running under certain well-defined conditions. This standard, though, was defined before the invention of the laser with its high degree of stability, and is not so good. A better definition of the metre can be made as the distance traveled by light in vacuum in a time of 1/2 997 924.58 (caesium clock) seconds. Thus we see that a unit of length can be defined either autonomously or in conjunction with the speed of light. The kilogram started as a lump of metal in Paris, but unlike its compatriot the metre continued in use in the form of carefully weighed copies. This was because Avogadro's number, which gives the number of atoms in a mass of material equal to the atomic number in grams, was not known by traditional means to very high precision. However, it is possible to obtain a better definition of the kilogram in terms of Avogadro's number derived from the lattice spacing of a pure crystal of a material like ^{26}Si, where the spacing can be determined by X-ray diffraction and optical interference. Thus, a unit of mass can be defined either primitively or in terms of the mass of a crystal of known size. We conclude that most accuracy can be achieved by defining a unit of time, and using this to define a unit of length, and then employing this to obtain a unit of mass. However, more direct definitions can be made for all of these quantities, and there is no reason as far as units are concerned why we should not absorb c, G and h.

This was actually realized by Planck, who noted that their base dimensions are such as to allow us to define 'natural' units of mass, length and time. (See Barrow 1983: similar units were actually suggested by Stoney somewhat earlier; and some workers have preferred to absorb $\hbar$

rather than h.) The correspondence between natural or Planck units and the conventional gram, centimetre and second can be summarized as follows:

$$1\ m_P \equiv \left(\frac{\hbar c}{G}\right)^{1/2} = 2.2 \times 10^{-5}\ g \qquad 1g = 4.6 \times 10^{4}\ m_P$$

$$1\ l_P \equiv \left(\frac{G\hbar}{c^3}\right)^{1/2} = 1.6 \times 10^{-33}\ cm \qquad 1cm = 6.3 \times 10^{32}\ l_P$$

$$1\ t_P \equiv \left(\frac{G\hbar}{c^5}\right)^{1/2} = 5.4 \times 10^{-44}\ s \qquad 1s = 1.9 \times 10^{43}\ t_P \quad .$$

In Planck units, all of the constants c, G and $\hbar$ become unity and they consequently disappear from the equations of physics.

This is convenient but it involves a choice of units only and does not necessarily imply anything more. It has often been stated that a consistent theory of quantum gravity that involves c, G and $\hbar$ would naturally produce particles of the Planck mass noted above. However, this is theoretically unjustified based on what we have discussed; and seems to be practically supported by the observation that the universe is not dominated by 10^{-5} g black holes. A more significant view is that all measurements and observations involve comparing one thing with another thing of similar type to produce what is ultimately a dimensionless number (see Dicke 1962; Bekenstein 1979; Barrow 1981; Smith 1983; Wesson 1992). The latter can have any value, and are the things that physics needs to explain. For example, the electromagnetic fine-structure constant $\alpha \equiv e^2 / \hbar c \cong 1/137$ needs to be explained, which is equivalent to saying that the electron charge needs to be explained (Griffiths 1987). The 'gravitational fine-structure constant' $Gm_p^2 / \hbar c \cong 5 \times 10^{-39}$ needs to be explained, which is equivalent to saying that the mass of the proton needs to be explained (Carr and Rees 1979). And along the same lines, we need to

explain the constant involved in the observed correlation between the spin angular momenta and masses of astronomical objects, which is roughly $GM^2/Jc \cong 1/300$ (Wesson 1983). In other words, we get no more out of dimensional analysis and a choice of units than is already present in the underlying equations, and neither technique is a substitute for proper physics.

The physics of explaining the charge of the electron or the mass of a proton, referred to above, probably lies in the future. However, some comments can be made now. As regards e, it is an observed fact that α is energy or distance-dependent. Equivalently, e is not a fundamental constant in the same class as c, h and G. The current explanation for this involves vacuum polarization, which effectively screens the charge of one particle as experienced by another (see Section 1.4). This mechanism is depressingly mechanical to some field theorists, and in attributing an active role to the vacuum would have been anathema to Einstein. [There are also alternative explanations for it, such as the influence of a scalar field, as discussed in Nodvik (1985) and Chapter 5.] However, the philosophy of trying to understand the electron charge, rather than just accepting it as a given, has undoubted merit. The same applies to the masses of the elementary particles, which however are unquantized and so present more of a challenge. The main question is not whether we wish to explain charges and masses, but rather what is the best approach.

In this regard, we note that both are geometrizable (Hoyle and Narlikar 1974; Wesson 1992). The rest mass of a particle m is the easiest to treat, since using G or h we can convert m to a length:

$$x_m \equiv \frac{Gm}{c^2} \qquad \text{or} \qquad x_m \equiv \frac{h}{mc} \ .$$

Physically, the choice here would conventionally be described as one between gravitational or atomic units, a ploy which has been used in several theories that deal with the nature of mass

(see Wesson 1978 for a review). Mathematically, the choice is one of coordinates, provided we absorb the constants and view mass as on the same footing as time and space (see Chapter 7). The electric charge of a particle q is harder to treat, since it can only be geometrized by including the gravitational constant via $x_q = \left(G/c^4\right)^{1/2} q$. This, together with the trite but irrefutable fact that masses can carry charges but not the other way round, suggests that mass is more fundamental than charge.

1.3 General Relativity

In the original form of this theory due to Einstein, space is regarded as a construct in which only the relations between objects have meaning. The theory agrees with all observations of gravitational phenomena, but the best books that deal with it are those which give a fair treatment of the theory's conceptual implications. Notably, those by Weinberg (1972), Misner, Thorne and Wheeler (1973), Rindler (1977) and Will (1993). We should also mention the book by Jammer (1961) on concepts of mass; and the conference proceedings edited by Barbour and Pfister (1995) on the idea due to Mach that mass locally depends on the distribution of matter globally. The latter was of course a major motivation for Einstein, and while not incorporated into standard general relativity is an idea that will reoccur in subsequent chapters.

The theory is built on 10 dimensionless potentials which are the independent elements in a 4 x 4 metric tensor $g_{\alpha\beta}$ (α,β = 0-3). These define the square of the distance between 2 nearby points in 4D via $ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta$. (Here a repeated index upstairs and downstairs is summed over, and below we will use the metric tensor to raise and lower indices on other tensors.) The coordinates x^α are in a local limit identified as $x^0 = ct$, $x^1 = x$, $x^2 = y$, $x^3 = z$ using Cartesians. However, because the theory employs tensors and therefore gives relations valid in any system

of coordinates (covariance), the space and time labels may be mixed up and combined arbitrarily. Thus space and time are not distinct entities. Also, the role of the speed of light c is to dimensionally transpose a quantity with the dimension T to one with dimension L, so that all 4 of x^α may be treated on the same footing. Partial derivatives with respect to the x^α can be combined to produce the Christoffel symbol $\Gamma^\gamma_{\alpha\beta}$, which enables one to create a covariant derivative such that the derivative of a vector is now given by $\nabla_\alpha V_\beta = \partial V_\alpha / \partial x^\beta - \Gamma^\gamma_{\alpha\beta} V_\gamma$. From $g_{\alpha\beta}$ and its derivatives, one can obtain the Ricci tensor $R_{\alpha\beta}$, the Ricci scalar R and the Einstein tensor $G_{\alpha\beta} \equiv R_{\alpha\beta} - R g_{\alpha\beta}/2$. The last is constructed so as to have zero covariant divergence: $\nabla_\alpha G^{\alpha\beta} = 0$. These tensors enable us to look at the relationship between geometry and matter. Specifically, the Einstein tensor $G_{\alpha\beta}$ can be coupled via a constant κ to the energy-momentum tensor $T_{\alpha\beta}$ that describes properties of matter: $G_{\alpha\beta} = \kappa T_{\alpha\beta}$. These are Einstein's field equations. In the weak-field limit where $g_{00} \cong \left(1 + 2\phi / c^2\right)$ for a fluid of density ρ, Einstein's equations give back Poisson's equation $\nabla^2 \phi = 4\pi G \rho$. This presumes that the coupling constant is $\kappa = 8\pi G / c^4$, and shows that Einstein gravity contains Newton gravity. However, Einstein's field equations have only been rigorously tested in the solar system and the binary pulsar, where the gravitational field exists essentially in empty space or vacuum. In this case, $T_{\alpha\beta} = 0$ and the field equations $G_{\alpha\beta} = 0$ are equivalent to the simpler set

$$R_{\alpha\beta} = 0 \qquad (\alpha, \beta = 0 - 3) \,. \tag{1.1}$$

These 10 relations serve in principle to determine the 10 $g_{\alpha\beta}$, and are the ones verified by observations.

Notwithstanding this, let us consider the full equations for a perfect isotropic fluid with density ρ and pressure p (i.e. there is no viscosity, and the pressure is equal in the 3 spatial directions). Then the energy-momentum tensor is $T_{\alpha\beta} = \left(p + \rho c^2\right) u_\alpha u_\beta - p g_{\alpha\beta}$ where u_α are the 4-velocities (see below). This is constructed so as to have zero divergence, and the equation of continuity and the equations of motion for the 3 spatial directions are derived from the 4 components of $\nabla_\alpha T^{\alpha\beta} = 0$. The covariant derivative here actually treats the metric tensor as a constant, so it is possible to add a term proportional to this to either the left-hand side or right-hand side of Einstein's equations. The former usage is traditional, so the full field equations are commonly written

$$R_{\alpha\beta} - \frac{R g_{\alpha\beta}}{2} + \Lambda g_{\alpha\beta} = \frac{8\pi G}{c^4}\left[\left(p + \rho c^2\right) u_\alpha u_\beta - p g_{\alpha\beta}\right]. \tag{1.2}$$

Here Λ is the cosmological constant, and its modulus is known to be small. It corresponds in the weak-field limit to a force per unit mass $|\Lambda| c^2 r / 3$ which increases with radius r from the centre of (say) the solar system, but is not observed to significantly affect the orbits of the planets. However, it could be insignificant locally but significant globally, as implied by its dimensions (L^{-2}). In this regard, it is instructive to move the Λ term over to the other side of the field equations and incorporate it into $T_{\alpha\beta}$ as a "vacuum" contribution to the density and pressure:

$$\rho_v = \frac{\Lambda c^2}{8\pi G} \qquad\qquad p_v = -\frac{\Lambda c^4}{8\pi G} \quad . \tag{1.3}$$

This "vacuum fluid" has the equation of state $p_v = -\rho_v c^2$, and while ρ_v is small by laboratory standards it could in principle be of the same order of magnitude as the material density of the galaxies (10^{-29} – 10^{-31} gm cm^{-3}). Also, while $|\Lambda|$ is constrained by general relativity and

observations of the present universe, there are arguments concerning the stability of the vacuum from quantum field theory which imply that it could have been larger in the early universe. But Λ (and G,c) are true constants in the original version of general relativity, so models of quantum vacuum transitions involve step-like phase changes (see e.g. Henriksen, Emslie and Wesson 1983). It should also be noted that while matter in the present universe has a pressure that is positive or close to zero ("dust"), there is in principle no reason why in the early universe or other exotic situations it cannot be taken negative. Indeed, any microscopic process which causes the particles of a fluid to attract each other can in a macroscopic way be described by $p < 0$ (the vacuum treated classically is a simple example). In fact, it is clear that p and ρ in general relativity are phenomenological, in the sense that they are labels for unexplained particle processes. It is also clear that the prime function of G and c is to dimensionally transpose matter labels such as p and ρ so that they match the geometrical objects of the theory.

The pressure and density are intimately connected to the motion of the fluid which they describe. This can be appreciated by looking at the general equation of motion, in the form derived by Raychaudhuri, and the continuity or conservation equation:

$$\ddot{R}\left(\frac{3}{R}\right) = 2\left(\omega^2 - \sigma^2\right) - \frac{4\pi G}{c^2}\left(3p + \rho c^2\right)$$

$$\dot{\rho}c^2 = -\left(p + \rho c^2\right)\frac{3\dot{R}}{R} \quad . \tag{1.4}$$

Here R is the scale factor of a region of fluid with vorticity ω, shear σ, and uniform pressure and density (see Ellis 1984: a dot denotes the total derivative with respect to time, and R should not be confused with the Ricci scalar introduced above and should not be taken as implying the existence of a physical boundary). From the first of (1.4) we see that the acceleration caused by a portion of the fluid depends on the combination $\left(3p + \rho c^2\right)$, so for mass to be attractive and

positive we need $(3p+\rho c^2)>0$. From the second of (1.4), we see that the rate of change of density depends on the combination $(p+\rho c^2)$, so for matter to be stable in some sense we need $(p+\rho c^2)>0$. These inequalities, sometimes called the energy conditions, should not however be considered sacrosanct. Indeed, gravitational energy is a slippery concept in general relativity, and there are several alternative definitions of "mass" (Hayward 1994). These go beyond the traditional concepts of active gravitational mass as the agent which causes a gravitational field, passive gravitational mass as the agent which feels it, and inertial mass as the agent which measures energy content (Bonnor 1989). What the above shows is that in a fluid-dynamical context, $(3p+\rho c^2)$ is the gravitational energy density and $(p+\rho c^2)$ is the inertial energy density.

For a fluid which is homogeneous and isotropic (≡ uniform), without vorticity or shear, Einstein's equations reduce to 2 relations commonly called after Friedmann:

$$8\pi G\rho = \frac{3}{R^2}\left(kc^2+\dot{R}^2\right)-\Lambda c^2,$$

$$\frac{8\pi Gp}{c^2} = -\frac{1}{R^2}\left(kc^2+\dot{R}^2+2\ddot{R}R\right)+\Lambda c^2 \quad . \tag{1.5}$$

Here k = ± 1, 0 is the curvature constant which describes the departure of the 3D part of spacetime from flat Minkowski (specified by $g_{\alpha\beta}=\eta_{\alpha\beta}=$ diagonal +1, -1, -1, -1). There are many solutions of (1.5) which are more or less in agreement with cosmological observations. The simplest is the Einstein-de Sitter model. It has k = 0, $\Lambda = 0$, p = 0, $\rho = 1/6\pi Gt^2$ and a scale factor R(t) which grows as $t^{2/3}$. However, it requires about 2 orders of magnitude more matter to be present than in the visible galaxies, a topic we will return to in Sections 1.6 and 4.2. In general, solutions of (1.5) are called Friedmann-Robertson-Walker (FRW), where the last two

names refer to the workers who derived the metric for these uniform cosmological models. This metric is commonly given in two different coordinate systems, whose justification has to do with whether one takes the global view wherein all directions in 3D space are treated the same, or the local view wherein quantities are measured from us as 'centre'. Noting that the radial coordinates r are different, the (3D) isotropic and non-isotropic forms of the metric are given by:

$$ds^2 = c^2dt^2 - \frac{R^2(t)}{\left(1+kr^2/4\right)^2}\left[dr^2 + r^2\,d\Omega^2\right]$$

$$ds^2 = c^2dt^2 - R^2(t)\left[\frac{dr^2}{\left(1-kr^2\right)} + r^2 d\Omega^2\right] \quad . \tag{1.6}$$

Here $d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\phi^2$ defines the angular part of the metric in spherical polar coordinates. A photon which moves radially in the field described by (1.6) is defined by ds = 0 with $d\theta = d\phi = 0$. Using the second of (1.6) its (coordinate-defined) velocity is then

$$\frac{dr}{dt} = \pm\frac{c\left(1-kr^2\right)^{1/2}}{R(t)} \quad . \tag{1.7}$$

Here the sign choice corresponds to whether the photon is moving towards or away from us. The important thing, though, is that the "speed" of the photon is <u>not</u> c.

This parameter, as noted in Section 1.2, is commonly regarded as defining an upper limit to the speed of propagation of causal effects. However, this interpretation is only true in the local, special-relativity limit. In the global, general-relativity case the size of causally-connected regions is defined by the concept of the horizon. An excellent account of this is given by its originator, Rindler (1977, p. 215). In the cosmological application, there are actually 2 kinds of horizon. An event horizon separates those galaxies we can see from those we cannot ever see even as $t \to \infty$; a particle horizon separates those galaxies we can see from those we cannot see

now at $t = t_0$ ($\cong 2 \times 10^{10}$ yr). FRW models exist which have both kinds of horizon, one but not the other, or neither. A model in the latter category is that of Milne. (It has $k = -1$, $\Lambda = 0$, $p = 0$ and R(t) proportional to t, and would solve the so-called horizon problem posed by the 3K microwave background did it not also have $\rho = 0$.) The distance to the particle horizon defines the size of that part of the universe which is in causal communication with us. The distance can be worked out quite simply for any k if we assume $\Lambda = p = 0$ (Weinberg 1972, p. 489). In terms of Hubble's parameter now $\left(H_0 \equiv \dot{R}_0 / R_0\right)$ and the deceleration parameter now $\left(q_0 = -\ddot{R}_0 R_0 / \dot{R}_0^2\right)$, the distances are given by:

$$d_{k=+1} = \frac{c}{H_0 (2q_0 - 1)^{1/2}} \cos^{-1}\left(\frac{1}{q_0} - 1\right), \qquad q_0 > \frac{1}{2}$$

$$d_{k=0} = \frac{2c}{H_0} = 3ct_0 \qquad , \qquad . \; q_0 = \frac{1}{2}$$

$$d_{k=-1} = \frac{c}{H_0 (1 - 2q_0)^{1/2}} \cosh^{-1}\left(\frac{1}{q_0} - 1\right), \qquad q_0 < \frac{1}{2} \; . \tag{1.8}$$

Even for the middle case, the Einstein-de Sitter model with flat 3-space sections, the distance to the horizon is not ct_0. This confirms what was noted above, and shows that in relativity the purpose of c is merely to transpose a time to a length.

Particles with finite as opposed to zero rest masses move not along paths with ds = 0 but along paths with s a minimum. In particle physics with a special-relativity metric, the action principle for the motion of a particle with mass m is commonly written $\delta\left[\int mds\right] = 0$. Assuming m = constant and replacing ds by its general relativity analog using $ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta$, the variation leads to 4 equations of motion:

$$\frac{du^\gamma}{ds} + \Gamma^\gamma_{\alpha\beta} u^\alpha u^\beta = 0 \quad . \tag{1.9}$$

This is the geodesic equation, and its 4 components serve in principle to determine the 4-velocities $u^\gamma \equiv dx^\gamma / ds$ as functions of the coordinates. We note that, in addition to the assumption that m is constant, m does not appear in (1.9): general relativity is not a theory of forces but a theory of accelerations. In practice, (1.9) can only be solved algebraically for certain solutions of the field equations. The latter in vacuum are (1.1), and we note here that these can be obtained from an action via $\delta\left[\int R(-g)^{1/2} d^4x\right] = 0$. Here g is the determinant of the metric tensor, which with the conventional split of spacetime into time and space has signature (+ - - -) so g is negative. The field equations with matter can also be obtained from an action, but split into a geometrical part and a matter part. However, the split of a metric into time and space parts, and the split of the field equations into geometric and matter parts, are to a certain extent subjective.

1.4 Particle Physics

This has evolved along different lines than gravitation, and while general relativity is monolithic, the standard model of particle physics is composite. Of relevance are the books by Ramond (1981), Griffiths (1987), and Collins, Martin and Squires (1989). The last is a good review of the connections between particle physics and cosmology, and also treats higher-dimensional theories of the types we will examine in subsequent sections. However, the present section is mainly concerned with standard 4D particle physics as based on Lagrangians, and the conceptual differences between gravitation and quantum theory.

The material is ordered by complexity: we consider the equations of Maxwell, Schrodinger, Klein-Gordon, Dirac, Proca and Yang-Mills; and then proceed to quantum chromodynamics and the standard model (including Glashow-Salam-Weinberg theory). As before, there is an emphasis on fundamental constants and the number of parameters required to make theory compatible with observation.

Classical electromagnetism is described by a 4-potential A_α and a 4-current J^α (covariant and contravariant quantities differ now by at most a sign). Then Maxwell's equations are contained in the tensor relations

$$\frac{\partial F^{\alpha\beta}}{\partial x^\alpha} = \frac{4\pi}{c} J^\beta, \qquad F_{\alpha\beta} \equiv \frac{\partial A^\beta}{\partial x^\alpha} - \frac{\partial A_\alpha}{\partial x^\beta}, \tag{1.10}$$

and the identities

$$\frac{\partial F_{\alpha\beta}}{\partial x^\gamma} + \frac{\partial F_{\beta\gamma}}{\partial x^\alpha} + \frac{\partial F_{\gamma\alpha}}{\partial x^\beta} = 0$$

implicit in the definition of the Faraday tensor $F_{\alpha\beta}$. However, Maxwell's equations may also be obtained by substituting the Lagrangian

$$L = -\frac{1}{16\pi} F^{\alpha\beta} F_{\alpha\beta} - \frac{1}{c} J^\alpha A_\alpha \tag{1.11}$$

in the Euler-Lagrange equations, which give (1.10). Strictly, L here is a Lagrangian density and has dimensions energy/volume, presuming we use the c.g.s./e.s.u. system of units. These units also imply that ε_0 does not appear (see Section 1.2). Thus c is the only constant that figures, in analogy with the original version of general relativity in which only G/c^4 figured (no cosmological constant). This is connected with the fact that these theories describe photons and gravitons with exactly zero rest mass.

Planck's constant $\hbar$ comes into the field theory of particles when the 3-momentum p and total energy E of a particle are replaced by space and time operators that act on a wave-function Ψ. Thus the prescriptions $p \rightarrow (\hbar / i)\nabla$ and $E \rightarrow (i\hbar)\partial / \partial t$ applied to the non-relativistic energy equation $p^2 / 2m + V = E$ (where m is rest mass and V is the potential energy) result in the Schrodinger equation

$$-\frac{\hbar^2}{2m}\nabla^2\Psi + V\Psi = i\hbar\frac{\partial\Psi}{\partial t} \ . \tag{1.12}$$

The path Lagrangian for this is L = T – V in general, which for a particle with charge q moving with a 3-velocity dx/dt << c in an electromagnetic field is $L = (m/2)(dx/dt)^2 - (q/c)A_\alpha dx^\alpha / dt$. The path action for this is $S = \int_1^2 L\,dt$, where the integral is between two points. The variation $\delta S = 0$ gives the equations of motion of the particle between these two points, which in classical theory is a unique path. In quantum theory, there are non-unique paths, but the sum over paths $\Sigma \exp(iS/\hbar)$ has the interpretation that the modulus squared is the probability that the particle goes from position 1 to 2. Clearly the phase $S/\hbar$ has to be dimensionless, and this is why $\hbar$ appears in the sum over paths. Instead of including it in the latter thing, however, we could instead use $\Sigma \exp(iS)$ and redefine the Lagrangian to be

$$L = \frac{m}{2\hbar}\left(\frac{dx}{dt}\right)^2 - \frac{q}{c\hbar}A_\alpha\frac{dx^\alpha}{dt} \ . \tag{1.13}$$

This has been pointed out by Hoyle and Narlikar (1974, p. 102; see also Ramond, 1981, p. 35). They go on to argue that since the second term in (1.13) contains another q implicit in A_α, it is the combination $q^2/\hbar$ that is important, and in it $\hbar$ can be absorbed into q^2. Also, in the first

term in (1.13) it is the combination $m/\hbar$ that is important, and in it $\hbar$ can be absorbed into m. Thus the Lagrangian reduces back to the form given before.

A similar prescription to that above applied to the relativistic energy equation $E^2 - p^2c^2 = m^2c^4$ or $p^\alpha p_\alpha = m^2c^2$ for a freely-moving particle (V = 0) results in the Klein-Gordon equation

$$-\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} + \nabla^2\phi = \left(\frac{mc}{\hbar}\right)^2\phi. \tag{1.14}$$

Here ϕ is a single scalar field and the Lagrangian is

$$L = \frac{1}{2}\left[\left(\frac{1}{c}\frac{\partial\phi}{\partial t}\right)^2 - (\nabla\phi)^2\right] - \frac{1}{2}\left(\frac{mc}{\hbar}\right)^2\phi^2 \ . \tag{1.15}$$

Equations (1.14) and (1.15) describe a spin-0 particle in flat spacetime. We will consider the generalization to curved spacetime below.

Spin-1/2 particles were described in another equation formulated by Dirac, who 'factorized' the energy relation $p^\alpha p_\alpha = m^2c^2$ with the help of four 4 x 4 matrices γ^α. These latter are related to the metric tensor of Minkowski spacetime by the relation $\gamma^\alpha\gamma^\beta + \gamma^\beta\gamma^\alpha = 2\eta^{\alpha\beta}$. The Dirac equation is

$$i\hbar\gamma^\alpha\frac{\partial\psi}{\partial x^\alpha} - mc\psi = 0. \tag{1.16}$$

Here ψ is a bi-spinor field, which can be thought of as a 4-element column matrix (though it is not a 4-vector) in which the upper two elements represent the two possible spin states of an electron while the lower two elements represent the two possible spin states of a positron. The Lagrangian is

$$L = i\hbar c\overline{\psi}\gamma^\alpha\frac{\partial\psi}{\partial x^\alpha} - mc^2\overline{\psi}\psi \ . \tag{1.17}$$

Here $\overline{\psi}$ is the adjoint spinor defined by $\overline{\psi} \equiv \psi^{+}\gamma^{0}$, where ψ^{+} is the usual Hermitian or transpose conjugate obtained by transposing ψ from a column to a row matrix and complex-conjugating its elements. The Lagrangian (1.17) is for a free particle. It is invariant under the global gauge or phase transformation $\psi \to e^{i\theta}\psi$ (where θ is any real number), because $\overline{\psi} \to e^{-i\theta}\overline{\psi}$ and the exponentials cancel out in the combination $\overline{\psi}\psi$. But it is not invariant under the local gauge transformation $\psi \to e^{i\theta(x)}\psi$ which depends on location in spacetime. If the principle of local gauge invariance is desired, it is necessary to replace (1.17) by

$$L = i\hbar c\overline{\psi}\gamma^{\alpha}\frac{\partial\psi}{\partial x^{\alpha}} - mc^{2}\overline{\psi}\psi - q\overline{\psi}\gamma^{\alpha}\psi A_{\alpha}. \tag{1.18}$$

Here A_{α} is a potential which we identify with electromagnetism and which changes under local gauge transformations according to $A_{\alpha} \to A_{\alpha} + \partial\lambda/\partial x^{\alpha}$ where $\lambda\left(x^{\alpha}\right)$ is a scalar function. In fact, we can say that the requirement of local gauge invariance for the Dirac Lagrangian (1.18) obliges the introduction of the field A_{α} typical of electromagnetism.

Actually the Lagrangian (1.18) should be even further extended by including a 'free' term for the gauge field. In this regard, the transformation $A_{\alpha} \to A_{\alpha} + \partial\lambda/\partial x^{\alpha}$ leaves $F_{\alpha\beta}$ unchanged, but not a term like $A^{\alpha}A_{\alpha}$. The appropriate term to add to (1.18) is therefore $\left(-1/16\pi\right)F^{\alpha\beta}F_{\alpha\beta}$, so the full Dirac Lagrangian is

$$L = i\hbar c\overline{\psi}\gamma^{\alpha}\frac{\partial\psi}{\partial x^{\alpha}} - mc^{2}\overline{\psi}\psi - \frac{1}{16\pi}F^{\alpha\beta}F_{\alpha\beta} - q\overline{\psi}\gamma^{\alpha}\psi A_{\alpha}. \tag{1.19}$$

If we define a current density $J^{\alpha} \equiv cq\left(\overline{\psi}\gamma^{\alpha}\psi\right)$, the last two terms give back Maxwell's Lagrangian (1.11). The Lagrange density (1.19) describes electrons or positrons interacting with an electromagnetic field consisting of massless photons. However, a term like the one we just

discarded $\left(A^{\alpha}A_{\alpha}\right)$ may be acceptable in a theory of massive gauge particles. Indeed, a field derived from a vector potential A_{α} associated with a particle of finite rest mass m is described by the Proca equation

$$\frac{\partial F^{\alpha\beta}}{\partial x^{\alpha}}+\left(\frac{mc}{\hbar}\right)^{2}A^{\beta}=0\ . \tag{1.20}$$

This describes a spin-1 particle such as a massive photon, and can be obtained from the Lagrangian

$$L=-\frac{1}{16\pi}F^{\alpha\beta}F_{\alpha\beta}+\frac{1}{8\pi}\left(\frac{mc}{\hbar}\right)^{2}A^{\alpha}A_{\alpha}\ . \tag{1.21}$$

Again we see the combination $m/\hbar$, so $\hbar$ may be absorbed here if so desired as it has been elsewhere.

If we consider two 4-component Dirac fields, it can be shown that a locally gauge-invariant Lagrangian can only be obtained if we introduce three vector fields $\left(A_{\alpha}^{1},A_{\alpha}^{2},A_{\alpha}^{3}\right)$. These can be thought of as a kind of 3-vector $\boldsymbol{A}_{\alpha}$. It is also necessary to change the definition of $F_{\alpha\beta}$ used above. The 3 components of the new quantity $\left(F_{\alpha\beta}^{1},F_{\alpha\beta}^{2},F_{\alpha\beta}^{3}\right)$ can again be thought of as a kind of vector, where now $\mathbf{F}_{\alpha\beta}\equiv\left[\partial\mathbf{A}_{\beta}/\partial x^{\alpha}-\partial\mathbf{A}_{\alpha}/\partial x^{\beta}-(2q/\hbar c)\left(\mathbf{A}_{\alpha}\times\mathbf{A}_{\beta}\right)\right]$. Further, the three Pauli matrices $\left(\tau_{1},\tau_{2},\tau_{3}\right)$ can be regarded as a vector τ. Then with dot products between vectors defined in the usual way, the Lagrangian is

$$L=i\hbar c\overline{\psi}\gamma^{\alpha}\frac{\partial\psi}{\partial x^{\alpha}}-mc^{2}\overline{\psi}\psi-\frac{1}{16\pi}\mathbf{F}^{\alpha\beta}\cdot\mathbf{F}_{\alpha\beta}-\left(q\overline{\psi}\gamma^{\alpha}\tau\psi\right)\cdot\mathbf{A}_{\alpha}\ . \tag{1.22}$$

Here ψ can be thought of as a column matrix with elements ψ_{1} and ψ_{2} , each of which is a 4-component Dirac spinor. The latter still describe spin-1/2 particles of mass m (where we have assumed both particles to have the same mass for simplicity), and they interact with three gauge

fields $A_\alpha^1, A_\alpha^2, A_\alpha^3$ which by gauge invariance must be massless. The kind of gauge invariance obeyed by (1.22) is actually more complex than that involving global and local phase transformations with $e^{i\theta}$ considered above. There ψ was a single spinor, whereas here ψ is a 2-spinor column matrix. This leads us to consider a 2 x 2 matrix which we take to be unitary ($U^+U = 1$). In fact the first two terms in (1.22) are invariant under the global transformation $\psi \to U\psi$, because $\overline{\psi} \to \overline{\psi}U^+$ so the combination $\overline{\psi}\psi$ is invariant. Just as any complex number of modulus 1 can be written as $e^{i\theta}$ with θ real, any unitary matrix can be written $U = e^{iH}$ with H Hermitian $\left(H^+ = H\right)$. Since H is a 2 x 2 matrix it involves 4 real numbers, say θ and a_1, a_2, a_3 which can be regarded as the components of a 3-vector **a**. As before, let τ be the 3-vector whose components are three 2 x 2 Pauli matrices, and let 1 stand for the 2 x 2 unit matrix. Then without loss of generality we can write $H = \theta 1 + \boldsymbol{\tau}\cdot\boldsymbol{a}$, so $U = e^{i\theta}e^{i\boldsymbol{\tau}\cdot\boldsymbol{a}}$. The first factor here is the old phase transformation. The second is a 2 x 2 unitary matrix which is special in that the determinant is actually 1. Thus $\psi \to e^{i\boldsymbol{\tau}\cdot\mathbf{a}}\psi$ is a global special-unitary 2-parameter, or SU(2), transformation. It should be recalled that this global invariance only involves the first two terms of the Lagrangian (1.22), which resemble the Lagrangian (1.17) of Dirac. The passage to local invariance along lines similar to those considered above leads to the other terms in the Lagrangian (1.22) and was made by Yang and Mills.

The full Yang-Mills Lagrangian (1.22) is invariant under local SU(2) gauge transformations, and leads to field equations that were originally supposed to describe two equal-mass spin-1/2 particles interacting with three massless spin-1 (vector) particles. In this form the theory is somewhat unrealistic, but still useful. For example, if we drop the first two terms in (1.22) we obtain a Lagrangian for the three gauge fields alone which leads to an interesting classical-type field theory that resembles Maxwell electrodynamics. This correspondence

becomes clear if like before we define currents $\mathbf{J}^{\alpha} \equiv cq\left(\overline{\psi}\gamma^{\alpha}\tau\psi\right)$, whereby the last two terms in (1.22) give a gauge-field Lagrangian

$$L = -\frac{1}{16\pi}\mathbf{F}^{\alpha\beta}\cdot\mathbf{F}_{\alpha\beta} - \frac{1}{c}\mathbf{J}^{\alpha}\cdot\mathbf{A}_{\alpha} \quad . \tag{1.23}$$

This closely resembles the Maxwell Lagrangian (1.11). But of course (1.23) gives rise to a considerably more complicated theory, solutions of which have been reviewed by Actor (1979). Some of these represent magnetic monopoles, which have not been observed. Some represent instantons and merons, which are hypothetical particles that tunnel between topologically distinct vacuum regions. Tunneling can in principle be important cosmologically. For example, Vilenkin (1982) has suggested that a certain type of instanton tunneling to de Sitter space from nothing can give birth to an inflationary universe. However, it is doubtful if the kinds of particles predicted by pure SU(2) Yang-Mills theory will ever have practical applications. The real importance of this theory is that it showed it was feasible to use a symmetry group involving non-commuting 2 x 2 matrices to construct a non-Abelian gauge theory. This idea led to more successful theories, notably one for the strong interaction based on SU(3) colour symmetry.

Quantum chromodynamics (QCD) is described by 3 coloured Dirac spinors that can be denoted $\psi_{\text{red}}, \psi_{\text{blue}}, \psi_{\text{green}}$ and 8 gauge fields given by a kind of 8-vector $\mathbf{A}_{\alpha}$. Each of $\psi_{\text{r}}, \psi_{\text{b}}, \psi_{\text{g}}$ is a 4-component Dirac spinor, and it is convenient to regard them as the elements of a column matrix ψ. This describes the colour states of a massive spin-1/2 quark. The 8 components of $\mathbf{A}_{\alpha}$ are associated with the 8 Gell-Mann matrices $\left(\lambda_{1-8}\right)$, which are the SU(3) equivalents of the Pauli matrices of SU(2), and describe massless spin-1 gluons. The Lagrangian for QCD can be constructed by adding together 3 Dirac Lagrangians like (1.17) above (one for each colour), insisting on local SU(3) gauge invariance (which brings in the 8 gauge fields), and

adding in a free gauge-field term (using $\mathbf{F}_{\alpha\beta}$ as defined above for the original Yang-Mills theory). The complete Lagrangian is

$$L = i\hbar c\overline{\psi}\gamma^{\alpha}\frac{\partial\psi}{\partial x^{\alpha}} - mc^{2}\overline{\psi}\psi - \frac{1}{16\pi}\mathbf{F}^{\alpha\beta}\cdot\mathbf{F}_{\alpha\beta} - \left(q\overline{\psi}\gamma^{\alpha}\lambda\psi\right)\cdot\mathbf{A}_{\alpha} \quad . \tag{1.24}$$

This resembles (1.22) above. However, the electric charge of a quark needs to be a fraction of e in order to account for the common hadrons as quark composites. And particle physics is best described by 6 quarks with different flavours (d, u, s, c, b, t) and different masses m. This means we really need 6 versions of (1.24) with different masses. A gluon does not carry electric charge, but it does carry colour charge. This is unlike its analogue the photon in electrodynamics, allowing bound gluon states (glueballs) and making chromodynamics generally quite complicated.

We do not need to go into the intricacies of QCD, especially since good reviews are available (Ramond 1981; Llewellyn Smith 1983; Griffiths 1987; Collins, Martin and Squires 1989). But a couple of points related to charges and masses are relevant to our discussion. In the case of electrons interacting via photons, the Dirac Lagrangian and the fact that $\alpha \equiv e^{2}/\hbar c \cong 1/137$ is small allows perturbation analysis to be used to produce very accurate models. Indeed, quantum electrodynamics (QED) gives predictions that are in excellent agreement with experiment. However, the coupling parameter whose asymptotic value is the traditional fine-structure constant is actually energy or distance dependent. As mentioned in Section 1.2, this is commonly ascribed to vacuum polarization. Thus, a positive charge (say) surrounded by virtual electrons and positrons tends to attract the former and repel the latter. (Virtual particles do not obey Heisenberg's uncertainty relation and in modern quantum field theory the vacuum is regarded as full of them.) There is therefore a screening process, which means that the effective value of the embedded charge (and α) increases as the distance

decreases. In analogy with QED, there is a similar process in QCD, but due to the different nature of the interaction the coupling parameter decreases as the distance decreases. This is the origin of asymptotic freedom, whose converse is that quarks in (say) a proton feel a strong restoring force if they move outwards and are in fact confined. In addition to the variable nature of coupling 'constants' and charges, the masses in QCD are also not what they appear to be. The m which appears in a Lagrangian like (1.24) is not really a given parameter, but is believed to arise from the spontaneous symmetry breaking which exists when a symmetry of the Lagrangian is not shared by the vacuum. Thus a manifestly symmetric Lagrangian with massless gauge-field particles can be rewritten in a less symmetric form by redefining the fields in terms of fluctuations about a particular ground state of the vacuum. This results in the gauge-field particles becoming massive and in the appearance of a massive scalar field or Higgs particle. In QCD, the quarks are initially taken to be massless, but if they have Yukawa-type couplings to the Higgs particle then they acquire masses. The Higgs mechanism in QCD, however, is really imported from the theory of the weak interaction, and has been mentioned here to underscore that the masses of the quarks are not really fundamental parameters.

The theory of the weak interaction was originally developed by Fermi as a way of accounting for beta decay, but is today mainly associated with Glashow, Weinberg and Salam who showed that it was possible to unify the weak and electromagnetic interactions (for reviews see Salam 1980 and Weinberg 1980). As it is formulated today, the theory of the weak interaction involves mediation by 3 very massive intermediate vector (spin-1) bosons, two of which ($W^{\pm}$) are electrically charged and one of which (Z^0) is neutral. These can be combined with the photon of electromagnetism via the symmetry group SU(2) $\otimes$ U(1), which is however spontaneously broken by the mechanism outlined in the preceding paragraph. Actually, the

massive Z^0 and the massless photon are combinations of states that depend on a weak mixing angle θ_w whose value is difficult to calculate from theory but is $\theta_w \cong 29^\circ$ from experiment. The theory of the weak interaction, like QED and QCD, involves a coupling parameter which is not constant.

What we have been discussing in the latter part of this section are parts of the standard model of particle physics, which symbolically unifies the electromagnetic, weak and strong interactions via the symmetry group U(1) $\otimes$ SU(2) $\otimes$ SU(3). An appealing feature of this theory is that with increasing energy the electromagnetic coupling increases while the weak and strong couplings decrease, suggesting that they come together at some unifying energy. This, however, is not known: it is probably of order 10^{16} GeV, but could be as large as the Planck mass of order 10^{19} GeV (see Weinberg 1983; Llewellyn Smith 1983; Ellis 1983; Kibble 1983; Griffiths 1987, p. 77; Collins, Martin and Squires 1989, p. 159). Also, there are uncertainties in the theory, notably to do with the QCD sector where the numbers of colors and flavors are conventionally taken as 3 and 6 respectively but could be different. This means that while in the conventional model there are 6 quark masses and 6 lepton masses, there could be more. In fact, if we include couplings and other things, there are at least 20 parameters in the theory (Ellis 1983). One might hope to reduce this by using a simple unifying group for U(1), SU(2) and SU(3), but the minimal example of SU(5) does not actually help much in this regard. And then there is the perennial question: What about gravity?

1.5 Kaluza-Klein Theory

The idea that the world may have more than 4 dimensions is due to Kaluza (1921), who with a brilliant insight realized that a 5D manifold could be used to unify Einstein's theory of

general relativity (Section 1.3) with Maxwell's theory of electromagnetism (Section 1.4). After some delay, Einstein endorsed the idea, but a major impetus was provided by Klein (1926). He made the connection to quantum theory by assuming that the extra dimension was microscopically small, with a size in fact connected via Planck's constant h to the magnitude of the electron charge e (Section 1.2). Despite its elegance, though, this version of Kaluza-Klein theory was largely eclipsed by the explosive development first of wave mechanics and then of quantum field theory. However, the development of particle physics led eventually to a resurgence of interest in higher-dimensional field theories as a means of unifying the long-range and short-range interactions of physics. Thus did Kaluza-Klein 5D theory lay the foundation for modern developments such as 11D supergravity and 10D superstrings (Section 1.6). In fact, there is some ambiguity in the scope of the phrase "Kaluza-Klein theory". We will mainly use it to refer to a 5D field theory, but even in that context there are several versions of it. The literature is consequently enormous, but we can mention the conference proceedings edited by De Sabbata and Schmutzer (1983), Lee (1984) and Appelquist, Chodos and Freund (1987). A recent comprehensive review of all versions of Kaluza-Klein theory is the article by Overduin and Wesson (1997a). The latter includes a short account of what is referred to by different workers as non-compactified, induced-matter or space-time-matter theory. Since this is the subject of the following chapters, the present section will be restricted to a summary of the main features of traditional Kaluza-Klein theory.

This theory is essentially general relativity in 5D, but constrained by two conditions. Physically, both have the motivation of explaining why we perceive the 4 dimensions of spacetime and (apparently) do not see the fifth dimension. Mathematically, they are somewhat different, however. (a) The so-called 'cylinder' condition was introduced by Kaluza, and

consists in setting all partial derivatives with respect to the fifth coordinate to zero. It is an extremely strong constraint that has to be applied at the outset of calculation. Its main virtue is that it reduces the algebraic complexity of the theory to a manageable level. (b) The condition of compactification was introduced by Klein, and consists in the assumption that the fifth dimension is not only small in size but has a closed topology (i.e. a circle if we are only considering one extra dimension). It is a constraint that may be applied retroactively to a solution. Its main virtue is that it introduces periodicity and allows one to use Fourier and other decompositions of the theory.

There are now 15 dimensionless potentials, which are the independent elements in a symmetric 5 x 5 metric tensor g_{AB} (A,B = 0-4: compare section 1.3). The first 4 coordinates are those of spacetime, while the extra one $x^4 = l$ (say) is sometimes referred to as the "internal" coordinate in applications to particle physics. In perfect analogy with general relativity, one can form a 5D Ricci tensor R_{AB}, a 5D Ricci scalar R and a 5D Einstein tensor $G_{AB} \equiv R_{AB} - Rg_{AB}/2$. The field equations would logically be expected to be $G_{AB} = kT_{AB}$ with some appropriate coupling constant k and a 5D energy-momentum tensor. But the latter is unknown, so from the time of Kaluza and Klein onward much work has been done with the 'vacuum' or 'empty' form of the field equations $G_{AB} = 0$. Equivalently, the defining equations are

$$R_{AB} = 0 \qquad (A, B = 0-4). \tag{1.25}$$

These 15 relations serve to determine the 15 g_{AB}, at least in principle.

In practice, this is impossible without some starting assumption about g_{AB}. This is usually connected with the physical situation being investigated. In gravitational problems, an assumption about $g_{AB} = g_{AB}(x^C)$ is commonly called a choice of coordinates, while in particle physics it is commonly called a choice of gauge. We will meet numerous concrete examples

later, where given the functional form of $g_{AB}(x^C)$ we will calculate the 5D analogs of the Christoffel symbols Γ^C_{AB} which then give the components of R_{AB} (Chapters 2-4). Kaluza was interested in electromagnetism, and realized that g_{AB} can be expressed in a form that involves the 4-potential A_α that figures in Maxwell's theory. He adopted the cylinder condition noted above, but also put g_{44} = constant. We will do a general analysis of the electromagnetic problem later (Chapter 5), but here we look at an intermediate case where $g_{AB} = g_{AB}(x^\alpha)$, $g_{44} = -\Phi^2(x^\alpha)$. This illustrates well the scope of Kaluza-Klein theory, and has been worked on by many people, including Jordan (1947, 1955), Bergmann (1948), Thiry (1948), Lessner (1982), and Liu and Wesson (1997). The coordinates or gauge are chosen so as to write the 5D metric tensor in the form

$$g_{AB} = \begin{bmatrix} \left(g_{\alpha\beta} - \kappa^2\Phi^2 A_\alpha A_\beta\right) & -\kappa\Phi^2 A_\alpha \\ -\kappa\Phi^2 A_\beta & -\Phi^2 \end{bmatrix} , \qquad (1.26)$$

where κ is a coupling constant. Then the field equations (1.25) reduce to

$$G_{\alpha\beta} = \frac{\kappa^2\Phi^2}{2} T_{\alpha\beta} - \frac{1}{\Phi}\left(\nabla_\alpha\nabla_\beta\Phi - g_{\alpha\beta}\Box\Phi\right)$$

$$\nabla^\alpha F_{\alpha\beta} = -3\frac{\nabla^\alpha\Phi}{\Phi} F_{\alpha\beta}$$

$$\Box\Phi = -\frac{\kappa^2\Phi^3}{4} F_{\alpha\beta}F^{\alpha\beta} \quad . \qquad (1.27)$$

Here $G_{\alpha\beta}$ and $F_{\alpha\beta}$ are the usual 4D Einstein and Faraday tensors (see sections 1.3 and 1.4 respectively), and $T_{\alpha\beta}$ is the energy-momentum tensor for an electromagnetic field given by $T_{\alpha\beta} = \left(g_{\alpha\beta}F_{\gamma\delta}F^{\gamma\delta}/4 - F_\alpha{}^\gamma F_{\beta\gamma}\right)/2$. Also $\Box \equiv g^{\alpha\beta}\nabla_\alpha\nabla_\beta$ is the wave operator, and the summation convention is in effect. Therefore we recognize the middle member of (1.27) as the 4 equations

of electromagnetism modified by a function, which by the last member of (1.27) can be thought of as depending on a wave-like scalar field. The first member of (1.27) gives back the 10 Einstein equations of 4D general relativity, but with a right-hand side which in some sense represents energy and momentum that are effectively derived from the fifth dimension. In short, Kaluza-Klein theory is in general a unified account of gravity, electromagnetism and a scalar field.

Kaluza's case $g_{44} = -\Phi^2 = -1$ together with the identification $\kappa = \left(16\pi G / c^4\right)^{1/2}$ makes (1.27) read

$$G_{\alpha\beta} = \frac{8\pi G}{c^4} T_{\alpha\beta}$$

$$\nabla^\alpha F_{\alpha\beta} = 0 \ . \tag{1.28}$$

These are of course the straight Einstein and Maxwell equations in 4D, but derived from vacuum in 5D, a consequence which is sometimes referred to as the Kaluza-Klein "miracle". However, these relations involve by (1.27) the choice of electromagnetic gauge $F_{\alpha\beta}F^{\alpha\beta} = 0$ and have no contribution from the scalar field. The latter could well be important, particularly in application to particle physics. In the language of that subject, the field equations (1.25) of Kaluza-Klein theory describe a spin-2 graviton, a spin-1 photon and a spin-0 boson which is thought to be connected with how particles acquire mass. The field equations can also be derived from a 5D action $\delta\left[\int R(-g)^{1/2} d^5x\right] = 0$, in a way analogous to what happens in 4D Einstein theory.

It is also possible to put Kaluza-Klein theory into formal correspondence with other 4D theories, notably the Brans-Dicke scalar-tensor theory (see Overduin and Wesson 1997a). This theory is sometimes cast in a form where the scalar field is effectively disguised by putting the functional dependence into G, the gravitational 'constant'. In this regard it belongs to a class of

4D theories, which includes ones by Dirac, Hoyle and Narlikar and Canuto et al., where the constants are allowed to vary with cosmic time (see Wesson 1978 and Barbour and Pfister 1995 for reviews). However, it should be stated with strength that Kaluza-Klein theory is essentially 5D, and trying to cast it into 4D form is technically awkward. It should also be noted that the reasons for treating 4D fundamental constants in this way are conceptually obscure.

1.6 Supergravity and Superstrings

These are based on the idea of supersymmetry, wherein each boson (integral spin) is matched with a fermion (half integral spin). Thus the particle which is presumed to mediate classical gravity (the graviton) has a partner (the gravitino). This kind of symmetry is natural, insofar as particle physics needs to account for both bosonic and fermionic matter fields. But it is also attractive because it leads to a cancellation of the enormous zero-point fields which otherwise exist but whose energy density is not manifested in the curvature of space (this is related to the so-called cosmological constant problem, which is discussed elsewhere). The literature on supergravity and superstrings is diverse, but we can mention the review articles by Witten (1981) and Duff (1996); and the books by West (1986) and Green, Schwarz and Witten (1987). The status of the electromagnetic zero-point field has been discussed by Wesson (1991). There is an obvious connection between 5D Kaluza-Klein theory, 11D supergravity and 10D superstrings. But while the former is more-or-less worked out, the latter are still in a state of development with an uncertain prognosis where it comes to their relevance to the real world. For this reason, and also because supersymmetry lies outside the scope of the rest of this work, we will content ourselves with a short history.

Supersymmetric gravity or supergravity began life as a 4D theory in 1976 but quickly made the jump to higher dimensions ("Kaluza-Klein supergravity"). It was particularly successful in 11D, for three principal reasons. First, Nahm showed that 11 was the maximum number of dimensions consistent with a single graviton (and an upper limit of two on particle spin). This was followed by Witten's proof that 11 was also the minimum number of dimensions required for a Kaluza-Klein theory to unify all the forces in the standard model of particle physics (i.e. to contain the gauge groups of the strong SU(3) and electroweak $SU(2)\otimes U(1)$ interactions). The combination of supersymmetry with Kaluza-Klein theory thus appeared to uniquely fix the dimensionality of the world. Second, whereas in lower dimensions one had to choose between several possible configurations for the matter fields, Cremmer et al. demonstrated in 1978 that in 11D there is a single choice consistent with the requirements of supersymmetry (in particular, that there be equal numbers of Bose and Fermi degrees of freedom). In other words, while a higher-dimensional energy-momentum tensor was still required, its form at least appeared somewhat natural. Third, Freund and Rubin showed in 1980 that compactification of the 11D model could occur in only two ways: to 7 or 4 compact dimensions, leaving 4 (or 7, respectively) macroscopic ones. Not only did 11D spacetime appear to be specially favored for unification, but it also split perfectly to produce the observed 4D world. (The other possibility, of a macroscopic 7D world, could however not be ruled out, and in fact at least one such model was constructed as well.) Buoyed by these three successes, 11D supergravity appeared set by the mid-1980s as a leading candidate for the hoped-for "theory of everything".

Unfortunately, certain difficulties have dampened this initial enthusiasm. For example, the compact manifolds originally envisioned by Witten (those containing the standard model)

turn out not to generate quarks or leptons, and to be incompatible with supersymmetry. Their most successful replacements are the 7-sphere and the "squashed" 7-sphere, described respectively by the symmetry groups SO(8) and SO(5) ⊗ SU(2). But these groups do not contain the minimum symmetry requirements of the standard model $[SU(3)\otimes SU(2)\otimes U(1)]$. This is commonly rectified by adding matter-related fields, the "composite gauge fields", to the 11D Lagrangian. Another problem is that it is very difficult to build chirality (necessary for a realistic fermion model) into an 11D theory. A variety of remedies have been proposed for this, including the common one of adding even more gauge fields, but none has been universally accepted. It should also be mentioned that supergravity theory is marred by a large cosmological constant in 4D, which is difficult to remove even by fine-tuning. Finally, quantization of the theory inevitably leads to anomalies.

Some of these difficulties can be eased by descending to 10 dimensions: chirality is easier to obtain, and many of the anomalies disappear. However, the introduction of chiral fermions leads to new kinds of anomalies. And the primary benefit of the 11D theory – its uniqueness – is lost: 10D is not specially favored, and the theory does not break down naturally into 4 macroscopic and 6 compact dimensions. (One can still find solutions in which this happens, but there is no reason why they should be preferred.) In fact, most 10D supergravity models not only require ad hoc higher-dimensional matter fields to ensure proper compactification, but entirely ignore gauge fields arising from the Kaluza-Klein mechanism (i.e. from symmetries of the compact manifold). A theory which requires all gauge fields to be effectively put in by hand can hardly be considered natural.

A breakthrough in solving the uniqueness and anomaly problems of 10D theory occurred when Green and Schwarz and Gross et al. showed that there were 2 (and only 2) 10D

supergravity models in which all anomalies could be made to vanish: those based on the groups SO(32) and $E_8 \otimes E_8$, respectively. Once again, extra terms (known as Chapline-Manton terms) had to be added to the higher-dimensional Lagrangian. This time, however, the addition was not completely arbitrary; the extra terms were those which would appear anyway if the theory were a low-energy approximation to certain kinds of supersymmetric string theory.

Supersymmetric generalizations of strings, or superstrings, are far from being understood. However, they have some remarkable virtues. For example, they retain the appeal of strings, wherein a point particle is replaced by an extended structure, which opens up the possibility of an anomaly-free approach to quantum gravity. (They do this while avoiding the generic prediction of tachyons, which plagued the old string theories.) Also, it is possible to make connections between certain superstring states and extreme black holes. (This may help resolve the problem of what happens to the information swallowed by classical singularities, which has been long standing in general relativity.) It is true that, for a while, there was thought to be something of a uniqueness problem for 10D superstrings, in that the groups SO(32) and $E_8 \otimes E_8$ admit five different string theories between them. But this difficulty was addressed by Witten, who showed that it is possible to view these five theories as aspects of a single underlying theory, now known as M-theory (for "Membrane"). The low-energy limit of this new theory, furthermore, turns out to be 11D supergravity. So it appears that the preferred dimensionality of spacetime may after all be 11, at least in regard to higher-dimensional theories which are compactified.

Supersymmetric particles such as gravitinos and neutralinos, if they exist, could provide the dark or hidden matter necessary to explain the dynamics of galaxies and bring cosmological observations into line with the simplest 4D cosmological models (see Section 1.3). However,

such 'dark' matter is probably not completely dark, because the particles concerned are unstable to decay in realistic (non-minimal) supersymmetric theories, and will contribute photons to the intergalactic radiation field. Observations of the latter can be used to constrain supersymmetric weakly interacting massive particles (WIMPs). Thus gravitinos and neutralinos are viable dark-matter candidates if they have decay lifetimes greater than of order 10^{11} yr and 10^{9} yr respectively (Overduin and Wesson 1997b). In this regard, they are favored over non-supersymmetric candidates such as massive neutrinos, axions and a possible decaying vacuum (Overduin and Wesson 1997c, 1992). There are other candidates, but clearly the identification of dark matter is an important way of testing supersymmetry.

1.7 Conclusion

This chapter has presented a potted account of theoretical physics as it exists at the present. We have learned certain things, namely: that fundamental constants are not (Section 1.2); that general relativity describes gravity excellently in curved 4D space (Section 1.3); that particle physics works well as a composite theory in flat 4D space (Section 1.4); that Kaluza-Klein theory in its original version unifies gravity and electromagnetism in curved 5D space (Section 1.5); and that supergravity and superstrings provide possible routes to new physics in 11D and 10D. So, where do we go from here?

There is no consensus answer to this, but let us consider the following line of reasoning. Physics is a description of the world as we perceive it (Eddington). In order to give a logical and coherent account of the maximum number of physical phenomena, we should presumably use the most advanced mathematical techniques. For the last century through to now, this implies that we should use geometry (Einstein, Riemann). The field equations of general relativity have no mathematical constraint as regards the number of dimensions in which they should be

applied: the choice follows from physics and depends on what we wish to explain. Also, there are certain ways of embedding lower-dimensional spaces with complicated structure in higher-dimensional spaces with simple structure, including flat ones (Campbell, Eisenhart: see the next chapter). So the question of how we can best describe gravity and particle physics is to a certain extent a question of algebraic technology. Now we might expect that the many quantum properties of elementary particles should be described by a space with a large number of dimensions. However, the classical properties of matter should be able to be handled by a space with a moderate number of dimensions.

The rest of this treatise is a compilation of (mainly technical) results which demonstrates this view. It will be seen that properties of matter such as the density and pressure of a fluid, as well as the rest mass and electric charge of a particle, can be derived from 5D geometry. This may sound surprising, but there are important differences between what we do now and what others have done before. The theory we will be working with is obviously not Einstein general relativity, since it is not 4D but 5D in nature. But it is not Kaluza-Klein either, because we do not invoke the hobbling cylinder condition typical of that theory, preferring instead to examine an unrestricted and rich 5D algebra. What we do in the following chapters also differs from previous work in that we do not need an explicit energy-momentum tensor: it will be seen that matter can be derived from geometry.

References (Chapter 1)

Actor, A. 1979, Rev. Mod. Phys. 51, 461.

Appelquist, T., Chodos, A., Freund, P.G.O. 1987, Modern Kaluza-Klein Theories (Addison-Wesley, Menlo Park).

Barbour, J.B., Pfister, H. (eds.) 1995, Mach's Principle: From Newton's Bucket to Quantum Gravity (Birkhauser, Boston).

Barrow, J.D. 1981, Quart. J. Roy. Astron. Soc. 22, 388.

Barrow, J.D. 1983, Quart. J. Roy. Astron. Soc. 24, 24.

Barrow, J.D., Tipler, F.J. 1986, The Anthropic Principle (Oxford U. Press, New York).

Bekenstein, J.D. 1979, Comm. Astrophys. 8, 89.

Bergmann, P. 1948, Ann. Math. (Princeton) 49, 255.

Bonnor, W.B. 1989, Gen. Rel. Grav. 21, 1143.

Bridgman, P.W. 1922, Dimensional Analysis (Yale U. Press, New Haven).

Bunge, M. 1971 in Problems in the Foundations of Physics (ed. Bunge, M., Springer, Berlin), 1.

Carr, B.J., Rees, M.J. 1979, Nature 278, 605.

Collins, P.D.B., Martin, A.D., Squires, E.J. 1989, Particle Physics and Cosmology (Wiley, New York).

De Sabbata, V., Schmutzer, E. (eds.) 1983, Unified Field Theories of More than 4 Dimensions Including Exact Solutions (World Scientific, Singapore).

Desloge, E.A. 1984, Am. J. Phys. 52, 312.

Dicke, R.H. 1962, Phys. Rev. 125, 2163.

Duff, M.J. 1996, Int. J. Mod. Phys. A11, 5623.

Eddington, A.S. 1929, Science and the Unseen World (MacMillan, New York).

Eddington, A.S. 1935, New Pathways in Science (Cambridge U. Press, Cambridge).

Eddington, A.S. 1939, The Philosophy of Physical Science (Cambridge U. Press, Cambridge).

Ellis, G.F.R. 1984, Ann. Rev. Astron. Astrophys. 22, 157.

Ellis, J. 1983, Phil. Trans. Roy. Soc. (London) A310, 279.

Green, M.B., Schwarz, J.H., Witten, E. 1987, Superstring Theory (Cambridge U. Press, Cambridge).

Griffiths, D. 1987, Introduction to Elementary Particles (Harper and Row, New York).

Hayward, S.A. 1994, Phys. Rev. D49, 831.

Henriksen, R.N., Emslie, A.G., Wesson, P.S. 1983, Phys. Rev. D27, 1219.

Hoyle, F., Narlikar, J.V. 1974, Action at a Distance in Physics and Cosmology (Freeman, San Francisco).

Jackson, J.D. 1975, Classical Electrodynamics (2nd ed., Wiley, New York).

Jammer, M. 1961, Concepts of Mass in Classical and Modern Physics (Harvard U. Press, Cambridge).

Jeffreys, H. 1973, Scientific Inference (3rd ed., Cambridge U. Press, Cambridge).

Jordan, P. 1947, Ann. Phys. (Leipzig) 1, 219.

Jordan, P. 1955, Schwerkraft und Weltall (Vieweg, Braunschweig).

Kaluza, T. 1921, Sitz. Preuss. Akad. Wiss. 33, 966.

Kibble, T.W.B. 1983, Phil. Trans. Roy. Soc. (London) A310, 293.

Klein, O. 1926, Z. Phys. 37, 895.

Lee, H.C. (ed.) 1984, An Introduction to Kaluza-Klein Theories (World Scientific, Singapore).

Lessner, G. 1982, Phys. Rev. D25, 3202.

Liu, H., Wesson, P.S. 1997, Class. Quant. Grav. 14, 1651.

Llewellyn Smith, C.H. 1983, Phil. Trans. Roy Soc. (London) A310, 253.

McCrea, W.H. 1986, Quart. J. Roy. Astron. Soc. 27, 137.

McCrea, W.H., Rees, M.J. (eds.) 1983, Phil Trans. Roy. Soc. (London) A310, 209.

Misner, C.W., Thorne, K.S., Wheeler, J.A. 1973, Gravitation (Freeman, San Francisco).

Nodvik, J.S. 1985, Phys. Rev. Lett. 55, 2519.

Overduin, J.M., Wesson, P.S. 1992, Vistas Astron. 35, 439.

Overduin, J.M., Wesson, P.S. 1997a, Phys. Reports 283, 303.

Overduin, J.M., Wesson, P.S. 1997b, Astrophys. J. 480, 470.

Overduin, J.M., Wesson, P.S. 1997c, Astrophys. J. 483, 77.

Petley, B.W. 1985, The Fundamental Constants and the Frontier of Measurement (Hilger, Bristol).

Ramond, P. 1981, Field Theory (Benjamin-Cummings, Reading).

Rindler, W. 1977, Essential Relativity (2nd ed., Springer, Berlin).

Salam, A. 1980, Rev. Mod. Phys. 52, 525.

Smith, K.F. 1983, Phil. Trans. Roy. Soc. (London) A310, 215.

Thiry, Y. 1948, Com. Rend. Acad. Sci. (Paris) 226, 216.

Vilenkin, A. 1982, Phys. Lett. B117, 25.

Weinberg, S. 1972, Gravitation and Cosmology (Wiley, New York).

Weinberg, S. 1980, Rev. Mod. Phys. 52, 515.

Weinberg, S. 1983, Phil. Trans. Roy. Soc. (London) A310, 249.

Wesson, P.S. 1978, Cosmology and Geophysics (Oxford U. Press-Hilger, New York).

Wesson, P.S. 1983, Astron. Astrophys. 119, 313.

Wesson, P.S. 1991, Astrophys J. 378, 466.

Wesson, P.S. 1992, Space Science Rev. 59, 365.

West, P. 1986, Introduction to Supersymmetry and Supergravity (World Scientific, Singapore).

Will, C.M. 1993, Theory and Experiment in Gravitational Physics (Cambridge U. Press, Cambridge).

Witten, E. 1981, Nucl. Phys. B 186, 412.

2. INDUCED-MATTER THEORY

"To make physics, the geometry should bite" (John Wheeler, Princeton, 1984)

2.1 Introduction

We import from the preceding chapter two important and connected ideas. First, as realized by several workers, the so-called fundamental constants like c, G and h have as their main purpose the transposition of physical dimensions. Thus, a mass can be regarded as a length; and physical quantities such as density and pressure can be regarded as having the same dimensions as the geometrical quantities that figure in general relativity. Second, physical quantities should be given a geometric interpretation, as envisaged by many people through time, including Einstein who wished to transmute the "base wood" of physics to the "marble" of geometry. An early attempt at this was made by Kaluza and Klein, who extended general relativity from 4 to 5 dimensions, but also applied severe restrictions to the geometry (the conditions of cylindricity and compactification). In this chapter, we will draw together results which have appeared in recent years which show that it is possible to interpret most properties of matter as the result of 5D Riemannian geometry, where however the latter allows dependence on the fifth coordinate and does not make assumptions about the topology of the fifth dimension.

This induced-matter theory has seen most work in 3 areas: (a) The case of uniform cosmological models is easiest to treat because of the high degree of symmetry involved, and is very instructive. (b) The soliton case is more complicated, but important because 5D solitons are the analogs of isolated 4D masses, and the 5D class of soliton solutions contains the unique 4D Schwarzschild solution. (c) The case of neutral matter can be treated quite generally, and lays the foundation for many applications where electromagnetic effects are not involved. After an outline of geometric feasibility (Section 2.2) we will give the main theoretical results in each of

the aforementioned areas (Sections 2.3, 2.4, 2.5). We defer the main observational implications to later chapters. Our conclusion (Section 2.6) will be that one extra dimension is enough to explain the phenomenological properties of classical matter.

From here on, we will absorb the fundamental constants c, G and h via a choice of units that renders their magnitudes unity. We will use the metric signature with diagonal = (+ - - - ±), where the last choice will be seen to depend on the physical application and not cause any problem with causality. Also, we will label 5D quantities with upper-case Latin letters (A = 0-4) and 4D quantities with lower-case Greek letters (α = 0-3). If there is a chance of confusion between the 4D part of a 5D quantity and the 4D quantity as conventionally defined, we will use a hat to denote the former and the straight symbol to denote the latter.

2.2 A 5D Embedding for 4D Matter

The 5D field equations for apparent vacuum in terms of the Ricci tensor are

$$R_{AB} = 0 \quad . \tag{2.1}$$

Equivalently, in terms of the 5D Ricci scalar and the 5D Einstein tensor $G_{AB} \equiv R_{AB} - Rg_{AB}/2$, they are

$$G_{AB} = 0 \quad . \tag{2.2}$$

By contrast, the 4D field equations with matter are given by Einstein's relations of general relativity:

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta} \quad . \tag{2.3}$$

The central thesis of induced-matter theory is that (2.3) are a subset of (2.2) with an effective or induced 4D energy-momentum tensor $T_{\alpha\beta}$ which contains the classical properties of matter.

That this is so will become apparent below when we treat several cases suggested by physics. However, it is also possible to approach the subject through algebra; and while results in the latter field were subsequent, they are general and can be summarized here. Thus, it is a direct consequence of a little-known theorem by Campbell that any analytic N-dimensional Riemannian manifold can be locally embedded in an (N+1)-dimensional Ricci-flat ($R_{AB} = 0$) Riemannian manifold (Romero, Tavakol and Zalaletdinov 1996). This is of great importance for establishing the generality of the proposal that 4D field equations with sources can be locally embedded in 5D field equations without sources. And it can be used to study lower-dimensional ($N < 4$) gravity, which may be easier to quantize than general relativity (Rippl, Romero and Tavakol 1995). It can also be employed to find new classes of 5D solutions (Lidsey et al. 1997). Some of the latter have the remarkable property that they are 5D flat but contain 4D subspaces that are curved and correspond to known physical situations (Wesson 1994; Abolghasem, Coley and McManus 1996). The latter do not, though, include the 4D Schwarszchild solution, which can only be embedded in a flat manifold with $N \geq 6$ (Schouten and Struik 1921; Tangherlini 1963; for general results in embeddings see Campbell 1926; Eisenhart 1949; Kramer et al. 1980). However, the principle is clear: curved 4D physics can be embedded in curved or flat 5D geometry, and we proceed to study 3 prime cases of this.

2.3 The Cosmological Case

There are many exact solutions known of (2.1) that are of cosmological type, meaning that the metric resembles that of Robertson-Walker and the dynamics is governed by equations like those of Friedmann (see Section 1.3). However, most of these do not involve dependence on the extra coordinate l and are from the induced-matter viewpoint very restricted. Thus while we will use one of these solutions below, we will concentrate on the much more significant

solutions of Ponce de Leon (1988). He found several classes of exact solutions of (2.1) whose metrics are separable and reduce to the standard 4D RW ones on the hypersurfaces l= constants. The induced matter and other properties associated with the most physical class of Ponce de Leon solutions were worked out by Wesson (1992a). Since then, many other cosmological solutions and their associated matter properties have been derived by various workers (see, e.g., Chatterjee and Sil 1993; Chatterjee, Panigrahi and Banerjee 1994; Liu and Wesson 1994; Liu and Mashhoon 1995; Billyard and Wesson 1996). In what follows, we will illustrate the transition from the 5D equations (2.1), (2.2) for apparent vacuum to the 4D equations (2.3) with matter, by using simple but realistic solutions.

It is convenient to consider a 5D metric with interval given by

$$dS^2 = e^{\nu}dt^2 - e^{\omega}\left(dr^2 + r^2 d\Omega^2\right) - e^{\mu}dl^2 \ . \tag{2.4}$$

Here the time coordinate x^0 = t and the space coordinates $x^{123} = r\theta\phi$ $\left(d\Omega^2 \equiv d\theta^2 + \sin^2\theta\, d\phi^2\right)$ have been augmented by the new coordinate $x^4 = l$. The metric coefficients ν, ω, and μ will depend in general on both t and l, partial derivatives with respect to which will be denoted by an overdot and an asterisk, respectively. Components of the Einstein tensor in mixed form are:

$$G_0^0 = e^{-\nu}\left(-\frac{3\dot{\omega}^2}{4} - \frac{3\dot{\omega}\dot{\mu}}{4}\right) + e^{-\mu}\left(\frac{3\overset{**}{\omega}}{2} + \frac{3\overset{*}{\omega}^2}{2} - \frac{3\overset{*}{\mu}\overset{*}{\omega}}{4}\right)$$

$$G_4^0 = e^{-\nu}\left(\frac{3\overset{**}{\omega}}{2} + \frac{3\dot{\omega}\overset{*}{\omega}}{4} - \frac{3\dot{\omega}\overset{*}{\nu}}{4} - \frac{3\overset{*}{\omega}\dot{\mu}}{4}\right)$$

$$G_1^1 = G_2^2 = G_3^3 = -e^{-\nu}\left(\ddot{\omega} + \frac{3\dot{\omega}^2}{4} + \frac{\ddot{\mu}}{2} + \frac{\dot{\mu}^2}{4} + \frac{\dot{\omega}\dot{\mu}}{2} - \frac{\dot{\nu}\dot{\omega}}{2} - \frac{\dot{\nu}\dot{\mu}}{4}\right)$$

$$+e^{-\mu}\left(\overset{\bullet\bullet}{\omega}+\frac{3\overset{\bullet}{\omega}^2}{4}+\frac{\overset{\bullet\bullet}{\nu}}{2}+\frac{\overset{\bullet}{\nu}^2}{4}+\frac{\overset{\bullet}{\omega}\overset{\bullet}{\nu}}{2}-\frac{\overset{\bullet}{\mu}\overset{\bullet}{\omega}}{2}-\frac{\overset{\bullet}{\nu}\overset{\bullet}{\mu}}{4}\right)$$

$$G_4^4=-e^{-\nu}\left(\frac{3\ddot{\omega}}{2}+\frac{3\dot{\omega}^2}{2}-\frac{3\dot{\nu}\dot{\omega}}{4}\right)+e^{-\mu}\left(\frac{3\overset{\bullet}{\omega}^2}{4}+\frac{3\overset{\bullet}{\omega}\overset{\bullet}{\nu}}{4}\right). \tag{2.5}$$

These are 5D components. We wish to match the terms in (2.5) with the components of the usual 4D perfect-fluid energy-momentum tensor. This is $T_{\alpha\beta}=(p+\rho)u_\alpha u_\beta-pg_{\alpha\beta}$, where $u^\alpha\equiv dx^\alpha/ds$, and for our case has components $T_0^0=\rho$, $T_1^1=-p$ for the density and pressure. Following the philosophy outlined above, we simply identify the new terms (due to the fifth dimension) in G_0^0 with ρ, and the new terms in G_1^1 with p. Then, collecting terms which depend on the new metric coefficient μ or derivatives with respect to the new coordinate l, we define

$$8\pi\rho\equiv-\frac{3}{4}e^{-\nu}\dot{\omega}\dot{\mu}+\frac{3}{2}e^{-\mu}\left(\overset{\bullet\bullet}{\omega}+\overset{\bullet}{\omega}^2-\frac{\overset{\bullet}{\mu}\overset{\bullet}{\omega}}{2}\right)$$

$$8\pi p\equiv e^{-\nu}\left(\frac{\ddot{\mu}}{2}+\frac{\dot{\mu}^2}{4}+\frac{\dot{\omega}\dot{\mu}}{2}-\frac{\dot{\nu}\dot{\mu}}{4}\right)-e^{-\mu}\left(\overset{\bullet\bullet}{\omega}+\frac{3\overset{\bullet}{\omega}^2}{4}+\frac{\overset{\bullet\bullet}{\nu}}{2}+\frac{\overset{\bullet}{\nu}^2}{4}+\frac{\overset{\bullet}{\omega}\overset{\bullet}{\nu}}{2}-\frac{\overset{\bullet}{\mu}\overset{\bullet}{\omega}}{2}-\frac{\overset{\bullet}{\nu}\overset{\bullet}{\mu}}{4}\right). \tag{2.6}$$

These are suggested identifications for 4D properties of matter in terms of 5D properties of geometry.

To see if they make physical sense to this point, we combine (2.6) and (2.5) with the field equations $G_B^A=0$ of (2.2). There comes

$$G_0^0 = -\frac{3}{4}e^{-\nu}\dot{\omega}^2 + 8\pi\rho = 0$$

$$G_4^0 = e^{-\nu}\left(\frac{3}{2}\ddot{\omega} + \frac{3}{4}\dot{\omega}\overset{*}{\omega} - \frac{3}{4}\dot{\omega}\overset{*}{\nu} - \frac{3}{4}\overset{*}{\omega}\dot{\mu}\right) = 0$$

$$G_1^1 = -e^{-\nu}\left(\ddot{\omega} + \frac{3}{4}\dot{\omega}^2 - \frac{\dot{\nu}\dot{\omega}}{2}\right) - 8\pi p = 0$$

$$G_4^4 = -e^{-\nu}\left(\frac{3}{2}\ddot{\omega} + \frac{3\dot{\omega}^2}{2} - \frac{3\dot{\nu}\dot{\omega}}{4}\right) + e^{-\mu}\left(\frac{3}{4}\overset{*}{\omega}^2 + \frac{3\overset{*}{\omega}\overset{*}{\nu}}{4}\right) = 0\ . \qquad (2.7)$$

We see from the first of these that ρ must be positive; and from the third that p could in principle be negative, as needed in classical descriptions of particle production in quantum field theory (see, e.g., Brout, Englert and Gunzig 1978; Guth 1981; and Section 1.3). To make further progress, however, we need explicit solutions of the field equations.

A simple solution of (2.1) or (2.2) that is well known but does not depend on l has $\nu = 0$, $\omega = \log t$, $\mu = -\log t$ in (2.4), which now reads

$$dS^2 = dt^2 - t\left(dr^2 + r^2 d\Omega^2\right) - t^{-1}dl^2. \qquad (2.8)$$

This has a shrinking fifth dimension, and from (2.6) or (2.7) density and pressure given by $8\pi\rho = 3/4t^2$, $8\pi p = 1/4t^2$. If these are combined to form the gravitational density $(\rho + 3p)$ and the proper radial distance $R \equiv e^{\omega/2}r$ is introduced, then the mass of a portion of the fluid is $M = 4\pi R^3(\rho + 3p)/3$. The field equations then ensure that $\ddot{R} = -M/R^2$ is obtained as usual for the law of motion. Similarly, the usual first law of thermodynamics is recovered by writing dE + pdV = 0 as $\left(\rho e^{3\omega/2}\right)^{\cdot} + p\left(e^{3\omega/2}\right)^{\cdot} = 0$ (E = energy, V = 3D volume; see Wesson 1992a). The equation of state of the fluid described by (2.8) is of course the $p = \rho/3$ typical of radiation.

To go beyond radiation, we use one of the classes of solutions to (2.1) or (2.2) due to Ponce de Leon (1988). With a redefinition of constants appropriate to the induced-matter theory, it has $e^{\nu} = l^2$, $e^{\omega} = t^{2/\alpha} l^{2/(1-\alpha)}$, $e^{\mu} = \alpha^2 (1-\alpha)^{-2} t^2$ in (2.4), which now reads

$$dS^2 = l^2 dt^2 - t^{2/\alpha} l^{2/(1-\alpha)} \left(dr^2 + r^2 d\Omega^2 \right) - \alpha^2 (1-\alpha)^{-2} t^2 dl^2 \,. \tag{2.9}$$

This has a growing fifth dimension, and density and pressure which depend on the one assignable constant α. From (2.6) or (2.7) they are

$$8\pi\rho = \frac{3}{\alpha^2 l^2 t^2} \qquad 8\pi p = \frac{(2\alpha - 3)}{\alpha^2 l^2 t^2} \,. \tag{2.10}$$

The presence of l here may appear puzzling at first, but the coordinates are of course arbitrary and the proper time is $T \equiv lt$. (Alternatively, the presence of $x^4 = l$ depends on whether we consider the pure 4D metric or the 4D part of the 5D metric.) In terms of this, $8\pi\rho = 3/\alpha^2 T^2$ and $8\pi p = (2\alpha - 3)/\alpha^2 T^2$. For $\alpha = 3/2$, $8\pi\rho = 4/3T^2$ and $p = 0$. While for $\alpha = 2$, $8\pi\rho = 3/4T^2$ and $8\pi p = 1/4T^2$. The former is identical to the 4D Einstein-de Sitter model for the late universe with dust. The latter is identical to the 4D standard model for the early universe with radiation or highly relativistic particles. (The coincidence of the properties of matter for this model with $\alpha = 2$ and the previous model does not necessarily imply that they are the same, since similar matter can belong to different solutions even in 4D.) As before, the usual forms of the law of motion and the first law of thermodynamics are recovered, provided we use the effective gravitational density of matter defined by the combination $\rho + 3p$ (see Section 1.3; these laws are recoverable generally for metrics with form (2.4) using the proper time $T \equiv e^{\nu/2} t$ and the proper distance $R \equiv e^{\omega/2} r$). The equation of state of the fluid described by (2.9) is $p = (2\alpha/3 - 1)\rho$, and so generally describes isothermal matter.

Other properties of (2.9) were studied by Wesson (1992a, 1994), including the sizes of horizons and the nature of the extra coordinate l. We defer further discussion of this model, because here we are mainly dealing with theoretical aspects of induced-matter theory. However, we note two things. First, the solutions (2.8) and (2.9) describe in general photons with zero rest mass and particles with finite rest mass, respectively; and the fact that the former does not depend on l whereas the latter does, gives us a first inkling that l is related to mass (see later). Second, the solution (2.9) gives an excellent description of matter in the late and early universe from the big-bang perspective of physics in 4D; but it has a somewhat amazing property from the perspective of geometry in 5D. Thus consider a coordinate transformation from t, r, l to T, R, L specified by

$$T=\left(\frac{\alpha}{2}\right)t^{1/\alpha}l^{1/(1-\alpha)}\left(1+\frac{r^2}{\alpha^2}\right)-\frac{\alpha}{2(1-2\alpha)}\left[t^{-1}l^{\alpha/(1-\alpha)}\right]^{(1-2\alpha)/\alpha}$$

$$R=rt^{1/\alpha}l^{1/(1-\alpha)}$$

$$L=\left(\frac{\alpha}{2}\right)t^{1/\alpha}l^{1/(1-\alpha)}\left(1-\frac{r^2}{\alpha^2}\right)+\frac{\alpha}{2(1-2\alpha)}\left[t^{-1}l^{\alpha/(1-\alpha)}\right]^{(1-2\alpha)/\alpha}. \tag{2.11}$$

Then (as may be verified by computer) the Ponce de Leon metric (2.9) in standard form becomes

$$dS^2=dT^2-\left(dR^2+R^2d\Omega^2\right)-dL^2\ . \tag{2.12}$$

This means that our universe can either be viewed as a 4D spacetime curved by matter or as a 5D flat space that is empty.

2.4 The Soliton Case

There is a class of exact solutions of (2.1) which has been rediscovered several times during the history of Kaluza-Klein theory. The metric is static, spherically-symmetric in ordinary (3D) space, and independent of the fifth coordinate. (There are many of these solutions

rather than one because Birkhoff's theorem does not apply in its conventional form in 5D.) The solutions have been interpreted as describing magnetic monopoles (Sorkin 1983), massive objects of which some called solitons have no gravitational effect (Gross and Perry 1983), and black holes (Davidson and Owen 1985). The first usage is questionable, because the 4D Schwarzschild solution is a special member of the class and gravitational in nature, and magnetic monopoles are in any case conspicuous by their absence in the real world. The last usage is misleading, because all but the Schwarzschild-like member of the class lack event horizons of the conventional sort. The middle usage can be extended, since in the induced-matter picture we will see that these solutions represent stable, extended objects (Wesson 1992b). Thus even though the word is over-worked in physics, we regard these 5D 1-body solutions as representing objects called solitons.

A particularly simple member of the soliton class was rediscovered by Chatterjee (1990). It is instructive to start with this, because it has been analyzed in standard or Schwarzschild-like coordinates (as opposed to the isotropic coordinates used below). Thus the 5D metric has an interval given by

$$dS^2 = \left[1 - \frac{2\sqrt{A}}{\sqrt{r^2 + A} + \sqrt{A}}\right] dt^2 - \frac{dr^2}{1 + A/r^2} - r^2 d\Omega^2 \quad - \left[1 - \frac{2\sqrt{A}}{\sqrt{r^2 + A} + \sqrt{A}}\right]^{-1} dl^2 \ . \quad (2.13)$$

Here the constant A would normally be identified in gravitational applications via the $r \to \infty$ limit as $\sqrt{A} = M_*$, the mass of an object like a star at the centre of ordinary space. However, as mentioned above, it cannot be assumed that (2.13) is a black hole. Indeed, the first metric coefficient in (2.13) goes to zero only for r tending to zero. In other words, the event horizon in the coordinates of (2.13) shrinks to a point at the centre of ordinary space. (This is not altered by a Killing-vector prescription for horizons and different sets of coordinates: see Wesson and

Ponce de Leon 1984.) To see what (2.13) actually represents, let us use the induced-matter approach. It is helpful to consider a metric we will come back to later, namely

$$dS^2 = e^{\nu}dt^2 - e^{\lambda}dr^2 - R^2 d\Omega^2 - e^{\mu}dl^2 . \tag{2.14}$$

This includes (2.13) if we assume that the metric coefficients ν, λ, R, μ depend only on the radius and not on the time or extra coordinate. Then following the same procedure as in the preceding section, we obtain the components of the induced 4D energy-momentum tensor :

$$\begin{aligned}
8\pi T_0^0 &= e^{-\lambda}\left(\frac{1}{2}\mu'' + \frac{1}{4}\mu'^2 + \frac{R'\mu'}{R} - \frac{1}{4}\lambda'\mu'\right) \\
8\pi T_1^1 &= e^{-\lambda}\left(\frac{R'\mu'}{R} + \frac{1}{4}\nu'\mu'\right) \\
8\pi T_2^2 &= e^{-\lambda}\left(\frac{1}{2}\mu'' + \frac{1}{4}\mu'^2 + \frac{R'\mu'}{2R} + \frac{1}{4}\nu'\mu' - \frac{1}{4}\lambda'\mu'\right) .
\end{aligned} \tag{2.15}$$

Here a prime denotes the partial derivative with respect to the radius. Other components are zero, and of course $T_3^3 = T_2^2$ because of spherical symmetry. The components (2.15) which define the properties of matter depend on derivatives of μ, that is upon the geometry of the fifth dimension. However, matter and geometry are unified via the field equations $G_B^A = 0$ of (2.2), and we can use these to rewrite (2.15) in a more algebraically convenient form:

$$\begin{aligned}
8\pi T_0^0 &= \frac{1}{R^2} - e^{-\lambda}\left(\frac{2R''}{R} + \frac{R'^2}{R^2} - \frac{R'\lambda'}{R}\right) \\
8\pi T_1^1 &= \frac{1}{R^2} - e^{-\lambda}\left(\frac{R'^2}{R^2} + \frac{R'\nu'}{R}\right) \\
8\pi T_2^2 &= -\frac{1}{4}e^{-\lambda}\left(2\nu'' + \nu'^2 + \frac{4R''}{R} - \frac{2R'\lambda'}{R} + \frac{2R'\nu'}{R} - \nu'\lambda'\right).
\end{aligned} \tag{2.16}$$

Substituting into these equations for the Chatterjee solution (2.13) gives

$$8\pi T_0^0 = \frac{A}{r^4}$$

$$8\pi T_1^1 = -\frac{2\sqrt{A}}{r^2\sqrt{r^2+A}} - \frac{2A\sqrt{A}}{r^4\sqrt{r^2+A}} - \frac{A}{r^4}$$

$$8\pi T_2^2 = \frac{\sqrt{A}}{r^2\sqrt{r^2+A}} + \frac{A\sqrt{A}}{r^4\sqrt{r^2+A}} \quad . \tag{2.17}$$

These components obey the equation of state

$$T_0^0 + T_1^1 + T_2^2 + T_3^3 = 0 \quad , \tag{2.18}$$

which is radiation-like. The matter described by (2.17) has a gravitational mass that can be evaluated using the standard 4D expression

$$M_g(r) \equiv \int \left(T_0^0 - T_1^1 - T_2^2 - T_3^3\right)\sqrt{-g_4}\, dV_3 \ , \tag{2.19}$$

where g_4 is the determinant of the 4-metric and dV_3 is a 3D volume element. Using (2.14) and (2.16) this gives

$$M_g(r) = \frac{1}{2}\int\left(\nu'' + \frac{1}{2}\nu'^2 + \frac{2R'\nu'}{R} - \frac{1}{2}\nu'\lambda'\right) e^{(\nu-\lambda)/2} R^2 dr. \tag{2.20}$$

This is most conveniently evaluated in the coordinates of the Chatterjee solution in the form (2.13). Thus putting $R \rightarrow r$ and integrating gives

$$M_g(r) = \frac{1}{2} r^2 e^{(\nu-\lambda)/2}\nu' \quad , \tag{2.21}$$

which with the coefficients of (2.13) is

$$M_g(r) = \sqrt{A}\left[\frac{\sqrt{r^2+A} - \sqrt{A}}{\sqrt{r^2+A} + \sqrt{A}}\right]^{1/2} \quad . \tag{2.22}$$

We see that $M_g(\infty) = \sqrt{A}$, agreeing with the usual metric-based definition of the mass as noted above. However, we also see that $M_g(0) = 0$, meaning that the gravitational mass goes to zero

at the centre. In summary, the Chatterjee soliton (2.13) is a ball of radiation-like matter whose density and pressure fall off very rapidly away from the centre, and whose integrated mass agrees with the conventional definition only at infinity.

The above concerned a special case of a broad class of 5D solutions which has been widely studied in forms due to Gross and Perry (1983) and Davidson and Owen (1985). These authors use different terminologies, particularly for two dimensionless constants which enter the solutions. The former use α, β and the latter use κ, ε where the two are related by $\kappa = -1/\beta$, $\varepsilon = -\beta/\alpha$. We adopt the latter notation, as it is more suited to the induced-matter approach. In it, positive effective density of matter requires $\kappa > 0$, and positive gravitational mass as measured at spatial infinity requires $\varepsilon\kappa > 0$ (see below). Thus, physicality requires that both κ and ε be positive. In terms of these constants the Chatterjee solution we have looked at already has just $\kappa = 1$, $\varepsilon = 1$. And the Schwarzschild solution we will look at below has $\varepsilon \to 0$, $\kappa \to \infty$, $\varepsilon\kappa \to 1$. We now proceed to consider the general class.

This has usually been discussed with the metric in spatially isotropic form, which we write as

$$dS^2 = e^{\nu}dt^2 - e^{\lambda}\left(dr^2 + r^2 d\Omega^2\right) - e^{\mu}dl^2 \ . \tag{2.23}$$

Then solutions of the apparently empty 5D field equations (2.1) or (2.2) are given by

$$e^{\nu/2} = \left(\frac{ar-1}{ar+1}\right)^{\varepsilon\kappa}$$

$$e^{\lambda/2} = \frac{(ar-1)(ar+1)}{a^2r^2}\left(\frac{ar+1}{ar-1}\right)^{\varepsilon(\kappa-1)}$$

$$e^{\mu/2} = \left(\frac{ar+1}{ar-1}\right)^{\varepsilon} \ . \tag{2.24}$$

Here a is a dimensional constant to do with the source, and the two dimensionless constants are related by a consistency relation derived from the field equations:

$$\varepsilon^2\left(\kappa^2-\kappa+1\right)=1\,. \tag{2.25}$$

This means that the class is a 2-parameter one, depending on a and one or the other of ε, κ. Also, we noted above that physicality requires that both κ and ε be positive. Now, the surface area of 2-shells around the centre of the 3-geometry varies as $(ar-1)^{1-\varepsilon(\kappa-1)}$, and will shrink to zero at $r = 1/a$ provided $1-\varepsilon(\kappa-1)>0$. This combined with (2.25) means $\kappa > 0$. That is, the centre of the 3-geometry is at $r = 1/a$ for physical choices of the parameters (see Billyard, Wesson and Kalligas 1995 for a more extensive discussion). Also, $e^{\nu}\rightarrow 0$ for $r \rightarrow 1/a$ for ε, $\kappa > 0$. So as for the Chatterjee case above, the event horizon for the general class shrinks to a point at the centre of ordinary space.

The properties of the induced matter associated with (2.24) can be worked out following the same procedure as before. The components of the induced 4D energy-momentum tensor are:

$$8\pi T_0^0=-e^{-\lambda}\left(\lambda''+\frac{1}{4}\lambda'^2+\frac{2\lambda'}{r}\right)$$

$$8\pi T_1^1=-e^{-\lambda}\left(\frac{1}{4}\lambda'^2+\frac{1}{2}\nu'\lambda'+\frac{\nu'}{r}+\frac{\lambda'}{r}\right)$$

$$8\pi T_2^2=-\frac{1}{2}e^{-\lambda}\left(\nu''+\lambda''+\frac{1}{2}\nu'^2+\frac{\nu'}{r}+\frac{\lambda'}{r}\right). \tag{2.26}$$

Substituting into these equations for the solutions in the form (2.24) and doing some tedious algebra gives

$$8\pi T_0^0=\frac{4\varepsilon^2\kappa a^6r^4}{(ar-1)^4(ar+1)^4}\left(\frac{ar-1}{ar+1}\right)^{2\varepsilon(\kappa-1)}$$

$$8\pi T_1^1=\frac{4\varepsilon a^5r^3}{(ar-1)^3(ar+1)^3}\left(\frac{ar-1}{ar+1}\right)^{2\varepsilon(\kappa-1)}-\frac{4\varepsilon a^6r^4(2\varepsilon+2ar-\varepsilon\kappa)}{(ar-1)^4(ar+1)^4}\left(\frac{ar-1}{ar+1}\right)^{2\varepsilon(\kappa-1)}$$

$$8\pi T_2^2 = -\frac{2\varepsilon a^5 r^3}{(ar-1)^3(ar+1)^3}\left(\frac{ar-1}{ar+1}\right)^{2\varepsilon(\kappa-1)} - \frac{4\varepsilon a^6 r^4(\varepsilon\kappa-\varepsilon-ar)}{(ar-1)^4(ar+1)^4}\left(\frac{ar-1}{ar+1}\right)^{2\varepsilon(\kappa-1)} \quad . \tag{2.27}$$

These components obey the same equation of state as before, namely $\left(T_0^0 + T_1^1 + T_2^2 + T_3^3\right) = 0$. If we average over the 3 spatial directions, this is equivalent to saying that the equation of state is $\bar{p} = \rho/3$. Also as before, we can calculate the standard 4D gravitational mass of a part of the fluid by using (2.19). This with (2.23) gives

$$M_g(r) = 4\pi \int \left(T_0^0 - T_1^1 - T_1^1 - T_3^3\right) e^{(\nu+3\lambda)/2} r^2 dr \ , \tag{2.28}$$

which with (2.26) is

$$M_g(r) = \frac{1}{2}\int\left(\nu'' + \frac{1}{2}\nu'^2 + \frac{2\nu'}{r} + \frac{1}{2}\nu'\lambda'\right) e^{(\nu+\lambda)/2} r^2 dr$$

$$= \frac{1}{2} r^2 e^{(\nu+\lambda)/2}\nu' \quad . \tag{2.29}$$

Then with the coefficients of (2.24) we obtain

$$M_g(r) = \frac{2\varepsilon k}{a}\left(\frac{ar-1}{ar+1}\right)^{\varepsilon} \quad . \tag{2.30}$$

This is the gravitational mass of a soliton as a function of (isotropic) radius r, and to be positive as measured at infinity requires that $\varepsilon\kappa > 0$. Since positive density requires that $\kappa > 0$ by (2.27) we see that we need both $\kappa > 0$ and $\varepsilon > 0$, as we stated above. Then (2.30) shows that $M_g(r = 1/a) = 0$, meaning again that the gravitational mass goes to zero at the centre. However, the mass as measured at spatial infinity is now $2\varepsilon\kappa/a$ and not just $2/a \equiv M$, as it was for the Chatterjee case.

This is interesting, and should be compared to what we obtain if we substitute parameters corresponding to the Schwarzschild case, namely $\varepsilon \to 0$, $\kappa \to \infty$, $\varepsilon\kappa \to 1$. Then (2.30) gives $M_g(r) =$ constant $= 2/a = M_*$. And the metric (2.23), (2.24) becomes

$$dS^2 = \left(\frac{1 - M_*/2r}{1 + M_*/2r}\right)^2 dt^2 - \left(1 + \frac{M_*}{2r}\right)^4 \left(dr^2 + r^2 d\Omega^2\right) - dl^2 . \qquad (2.31)$$

This is just the Schwarzschild solution (in isotropic coordinates) plus a flat and therefore physically innocuous extra dimension. In other words, if we use the conventionally defined 4D gravitational mass as a diagnostic for 5D solitons, we recover the usual 4D Schwarzschild mass exactly.

We have carried out a numerical investigation of preceding relations to clarify the status of the Schwarzschild solution (Wesson and Ponce de Leon 1994). The problem is that if we set $\varepsilon = 0$ and $\varepsilon\kappa = 1$ then (2.30) gives $M_g(r) = 2/a$ for all r; but if we keep ε small and let $r \to 1/a$, then (2.30) gives $M_g\,(r = 1/a) = 0$ irrespective of ε. Clearly the limit by which one is supposed to recover the Schwarzschild solution from the soliton solutions is ambiguous. However, our numerical results show that, from the viewpoint of perturbation analysis at least, the Schwarzschild case is just a highly compressed soliton. We have also looked at other definitions for the mass of a soliton, including the so-called proper mass (which depends on an integral involving T_0^0 only and is badly defined at the centre) and the ADM mass (which depends on a product of field strength and area and is well defined at the centre). To clarify what happens near the centre of a soliton, we have also calculated the geometric scalars for metric (2.23), (2.24). The relevant 5D invariant is the Kretschmann scalar $K \equiv R_{ABCD}R^{ABCD}$, which we have evaluated algebraically and checked by computer. It is

$$K = \frac{192a^{10}r^6}{(a^2r^2-1)^8}\left(\frac{ar-1}{ar+1}\right)^{4\varepsilon(\kappa-1)}\left\{1-2\varepsilon(\kappa-1)(2+\varepsilon^2\kappa)ar\right.$$

$$\left.+2(3-\varepsilon^4\kappa^2)a^2r^2-2\varepsilon(\kappa-1)(2+\varepsilon^2\kappa)a^3r^3+a^4r^4\right\}. \qquad (2.32)$$

Taking into account the constraint (2.25), this may be found to diverge for $\kappa > 0$ at $r = 1/a$, showing that there is a geometric singularity at the centre as we have defined the latter. [Note that if we let $\varepsilon \to 0$, $\kappa \to \infty$, $\varepsilon\kappa \to 1$ then (2.32) gives $K = 192a^{10}r^6/(ar+1)^{12}$ in isotropic coordinates, or $K = 48M_*^2/(r')^6$ in curvature coordinates where $r' = r(1+1/ar)^2$ and $a = 2/M_*$. This result agrees formally with the one from Einstein theory, but from our viewpoint no longer has much significance since the point $r' = 0$ or $r = -1/a$ is not part of the manifold, which ends at $r = 1/a$ or $r' = 2M_*$.] The relevant 4D invariant is $C \equiv R_{\alpha\beta}R^{\alpha\beta}$, which is most easily evaluated algebraically using the field equations. The latter are (2.1) or $R_{AB} = 0$, but if there is no dependency on the extra coordinate read just $R_{\alpha\beta} = 8\pi T_{\alpha\beta}$ because the 4D Ricci scalar R is zero. Then $C = 64\pi^2\left[\left(T_0^0\right)^2+\left(T_1^1\right)^2+2\left(T_2^2\right)^2\right]$, and can be evaluated using the components of the energy-momentum tensor (2.27). It is

$$C = \frac{8\varepsilon^2a^{10}r^6}{(a^2r^2-1)^8}\left(\frac{ar-1}{ar+1}\right)^{2\varepsilon(\kappa-1)}\left\{3+4\varepsilon(3-2\kappa)ar\right.$$

$$\left.+2(3+6\varepsilon^2+4\varepsilon^2\kappa^2-8\varepsilon^2\kappa)a^2r^2+12\varepsilon a^3r^3+3a^4r^4\right\}. \qquad (2.33)$$

This also diverges at $r = 1/a$, confirming that there is a singularity in the geometry at the centre. In conjunction with the fact that most of them do not have event horizons of the standard sort, this means that technically solitons should be classified as naked singularities.

Whether or not we can see to the centre of a soliton, practically, is a different question. What was a point mass in 4D general relativity has become a finite object in 5D induced-matter theory. The fluid is 'hot', with anisotropic pressure and density that falls off rapidly away from

the centre (for large distances it goes as M_*^2 / r^4 where M_* is the mass as measured at spatial infinity). The Schwarzschild solution is somewhat anomalous, but can be regarded as a soliton where matter is so concentrated towards the centre as to leave most of space empty. In short, solitons are 'holes' in the geometry surrounded by induced matter.

2.5 The Case of Neutral Matter

Kaluza-Klein theory in 5D has traditionally identified the $g_{4\alpha}$ components of the metric tensor with the potentials A_α of classical electromagnetism (see Section 1.5). These set to zero therefore give in some sense a description of neutral matter. However, any fully covariant 5D theory, such as induced-matter theory, has 5 coordinate degrees of freedom, which used judiciously can lead to considerable algebraic simplification without loss of generality. Therefore, a natural case to study is specified by $g_{4\alpha} = 0$, $g_{44} \neq 0$. This removes the explicit electromagnetic potentials and leaves one coordinate degree of freedom over to be used appropriately (e.g., to simplify the equation of motion of a particle). This choice of coordinates or choice of gauge involves $g_{\alpha\beta} = g_{\alpha\beta}(x^A)$, $g_{44} = g_{44}(x^A)$ and so is not restricted by the cylinder condition of old Kaluza-Klein theory. It admits fluids consisting of particles with finite or zero rest mass, and thus includes the cases we have studied in Sections 2.3 and 2.4. In the present section, we follow Wesson and Ponce de Leon (1992). Our aims are to give a reasonably self-contained account of the matter gauge and to lay the foundation for later applications.

We have a 5D interval $dS^2 = g_{AB}dx^A dx^B$ where

$$g_{\alpha\beta} = g_{\alpha\beta}(x^A) \qquad g_{4\alpha} = 0$$

$$g_{44} \equiv \varepsilon\Phi^2(x^A) \qquad g_{44} = \frac{1}{g_{44}} = \frac{\varepsilon}{\Phi^2} \quad . \tag{2.34}$$

Here $\varepsilon^2 = 1$ and the signature of the scalar part of the metric is left general. (We will see later that there are well-behaved classical solutions of the field equations with $\varepsilon = +1$ as well as the often-assumed $\varepsilon = -1$, and the freedom to choose this may also help with the Euclidean approach to quantum gravity.) The 5D Ricci tensor in terms of the 5D Christoffel symbols is given by

$$R_{AB} = \left(\Gamma^C_{AB}\right)_{,C} - \left(\Gamma^C_{AC}\right)_{,B} + \Gamma^C_{AB}\Gamma^D_{CD} - \Gamma^C_{AD}\Gamma^D_{BC} \ . \tag{2.35}$$

Here a comma denotes the partial derivative, and below we will use a semicolon to denote the ordinary (4D) covariant derivative. Putting $A \rightarrow \alpha$, $B \rightarrow \beta$ in (2.35) gives us the 4D part of the 5D quantity. Expanding some summed terms on the r.h.s. by letting $C \rightarrow \lambda,4$ etc. and rearranging gives

$$\begin{aligned}\hat{R}_{\alpha\beta} = &\left(\Gamma^\lambda_{\alpha\beta}\right)_{,\lambda} + \left(\Gamma^4_{\alpha\beta}\right)_{,4} - \left(\Gamma^\lambda_{\alpha\lambda}\right)_{,\beta} - \left(\Gamma^4_{\alpha 4}\right)_{,\beta} \\ &+\Gamma^\lambda_{\alpha\beta}\Gamma^\mu_{\lambda\mu} + \Gamma^\lambda_{\alpha\beta}\Gamma^4_{\lambda 4} + \Gamma^4_{\alpha\beta}\Gamma^D_{4D} - \Gamma^\mu_{\alpha\lambda}\Gamma^\lambda_{\beta\mu} - \Gamma^4_{\alpha\lambda}\Gamma^\lambda_{\beta 4} - \Gamma^D_{\alpha 4}\Gamma^4_{\beta D} \ .\end{aligned} \tag{2.36}$$

Part of this is the conventional Ricci tensor that only depends on indices 0123, so

$$\hat{R}_{\alpha\beta} = R_{\alpha\beta} + \left(\Gamma^4_{\alpha\beta}\right)_{,4} - \left(\Gamma^4_{\alpha 4}\right)_{,\beta} + \Gamma^\lambda_{\alpha\beta}\Gamma^4_{\lambda 4} + \Gamma^4_{\alpha\beta}\Gamma^D_{4D} - \Gamma^4_{\alpha\lambda}\Gamma^\lambda_{\beta 4} - \Gamma^D_{\alpha 4}\Gamma^4_{\beta D} \ . \tag{2.37}$$

To evaluate this we need the Christoffel symbols.

These can be tabulated here in appropriate groups:

$$\Gamma^4_{\alpha\beta} = -\frac{g^{44}\overset{*}{g}_{\alpha\beta}}{2} \qquad \Gamma^4_{\alpha 4} = \frac{g^{44}g_{44,\alpha}}{2}$$

$$\Gamma^D_{4D} = \frac{g^{DC}\overset{*}{g}_{DC}}{2} \qquad \Gamma^\lambda_{\beta 4} = \frac{g^{\lambda C}\overset{*}{g}_{\beta C}}{2}$$

$$\Gamma^D_{\alpha 4} = \frac{g^{D4}g_{44,\alpha}}{2} + \frac{g^{D\gamma}\overset{*}{g}_{\alpha\gamma}}{2} \qquad \Gamma^4_{\beta D} = \frac{g^{44}g_{D4,\beta}}{2} - \frac{g^{44}g_{\beta D,4}}{2} \ . \tag{2.38}$$

$$\Gamma^{\lambda}_{44} = -\frac{g^{\lambda\beta}g_{44,\beta}}{2} \qquad \Gamma^{\lambda}_{4\lambda} = \frac{g^{\lambda\beta}\overset{*}{g}_{\lambda\beta}}{2}$$

$$\Gamma^{\mu}_{\lambda\mu} = \frac{g^{\mu\beta}g_{\mu\beta,\lambda}}{2} \qquad \Gamma^{4}_{44} = \frac{g^{44}\overset{*}{g}_{44}}{2}$$

$$\Gamma^{\lambda}_{4\mu} = \frac{g^{\lambda\beta}\overset{*}{g}_{\mu\beta}}{2} \qquad \Gamma^{4}_{4\mu} = \frac{g^{44}g_{44,\mu}}{2} \quad . \tag{2.39}$$

$$\Gamma^{\lambda}_{4\alpha} = \frac{g^{\lambda\mu}\overset{*}{g}_{\alpha\mu}}{2} \qquad \Gamma^{4}_{4\alpha} = \frac{g^{44}g_{44,\alpha}}{2}$$

$$\Gamma^{\lambda}_{4\lambda} = \frac{g^{\lambda\beta}\overset{*}{g}_{\lambda\beta}}{2} \qquad \Gamma^{4}_{44} = \frac{g^{44}g_{44,4}}{2}$$

$$\Gamma^{\mu}_{\lambda\mu} = \frac{g^{\mu\nu}g_{\mu\nu,\lambda}}{2} \qquad \Gamma^{\mu}_{\alpha\lambda} = \frac{g^{\mu\sigma}}{2}\left(g_{\lambda\sigma,\alpha} + g_{\sigma\alpha,\lambda} - g_{\alpha\lambda,\sigma}\right)$$

$$\Gamma^{\lambda}_{44} = \frac{-g^{\alpha\lambda}g_{44,\alpha}}{2} \qquad \Gamma^{4}_{\alpha\lambda} = -\frac{g^{44}\overset{*}{g}_{\alpha\lambda}}{2} \quad . \tag{2.40}$$

We will use these respectively to evaluate (2.37) above and (2.44), (2.54) below.

Thus substituting into and expanding some terms in (2.37) gives

$$\hat{R}_{\alpha\beta} = R_{\alpha\beta} - \frac{\overset{*}{g}^{44}\overset{*}{g}_{\alpha\beta}}{2} - \frac{g^{44}\overset{**}{g}_{\alpha\beta}}{2} - \frac{g^{44}{}_{,\beta}g_{44,\alpha}}{2} - \frac{g^{44}g_{44,\alpha\beta}}{2}$$

$$+\frac{g^{44}g_{44,\lambda}\Gamma^{\lambda}_{\alpha\beta}}{2} - \frac{g^{\mu\nu}\overset{*}{g}_{\mu\nu}g^{44}\overset{*}{g}_{\alpha\beta}}{4} - \frac{(g^{44})^2\overset{*}{g}_{\alpha\beta}\overset{*}{g}_{44}}{4} + \frac{g^{\lambda\mu}g^{44}\overset{*}{g}_{\alpha\lambda}\overset{*}{g}_{\beta\mu}}{2} - \frac{(g^{44})^2 g_{44,\alpha}g_{44,\beta}}{4} \quad . \tag{2.41}$$

Some of the terms here may be rewritten using

$$-\frac{g^{44}{}_{,\beta}g_{44,\alpha}}{2} - \frac{g^{44}g_{44,\alpha\beta}}{2} + \frac{g^{44}g_{44,\lambda}\Gamma^{\lambda}_{\alpha\beta}}{2} - \frac{(g^{44})^2 g_{44,\alpha}g_{44,\beta}}{4}$$

$$= -\frac{1}{\Phi}\left(\Phi_{\alpha;\beta} - \Phi_{\lambda}\Gamma^{\lambda}_{\alpha\beta}\right) \equiv -\frac{\Phi_{\alpha;\beta}}{\Phi} \quad , \tag{2.42}$$

where $\Phi_\alpha \equiv \Phi_{,\alpha}$. Then (2.41) gives

$$\hat{R}_{\alpha\beta} = R_{\alpha\beta} - \frac{\Phi_{\alpha;\beta}}{\Phi} + \frac{\varepsilon}{2\phi^2}\left(\frac{\overset{*}{\Phi}\overset{*}{g}_{\alpha\beta}}{\Phi} - \overset{**}{g}_{\alpha\beta} + g^{\lambda\mu}\overset{*}{g}_{\alpha\lambda}\overset{*}{g}_{\beta\mu} - \frac{g^{\mu\nu}\overset{*}{g}_{\mu\nu}\overset{*}{g}_{\alpha\beta}}{2}\right). \tag{2.43}$$

We will use this below when we consider the field equations.

Returning to (2.35), we put $A = 4$, $B = 4$ and expand with $C \to \lambda$, 4 etc. to obtain

$$R_{44} = \left(\Gamma^\lambda_{44}\right)_{,\lambda} - \left(\Gamma^\lambda_{4\lambda}\right)_{,4} + \Gamma^\lambda_{44}\Gamma^\mu_{\lambda\mu} + \Gamma^4_{44}\Gamma^\mu_{4\mu} - \Gamma^\lambda_{4\mu}\Gamma^\mu_{4\lambda} - \Gamma^4_{4\mu}\Gamma^\mu_{44}\ . \tag{2.44}$$

The Christoffel symbols here are tabulated in (2.39), and cause (2.44) to become

$$R_{44} = -\frac{g^{\lambda\beta}{}_{,\lambda}g_{44,\beta}}{2} - \frac{g^{\lambda\beta}g_{44,\beta\lambda}}{2} - \frac{\overset{*}{g}^{\lambda\beta}\overset{*}{g}_{\lambda\beta}}{2} - \frac{g^{\lambda\beta}\overset{**}{g}_{\lambda\beta}}{2}$$

$$-\frac{g^{\lambda\beta}g_{44,\beta}g^{\mu\sigma}g_{\mu\sigma,\lambda}}{4} + \frac{g^{44}\overset{*}{g}_{44}g^{\lambda\beta}\overset{*}{g}_{\lambda\beta}}{4} - \frac{g^{\mu\beta}\overset{*}{g}_{\lambda\beta}g^{\lambda\sigma}\overset{*}{g}_{\mu\sigma}}{4} + \frac{g^{44}g_{44,\lambda}g^{\lambda\beta}g_{44,\beta}}{4}\ . \tag{2.45}$$

Some of the terms here may be rewritten using

$$-\frac{g^{\lambda\beta}g_{44,\beta}}{2} - \frac{g^{\lambda\beta}g_{44,\beta\lambda}}{2} - \frac{g^{\lambda\beta}g_{44,\beta}g^{\mu\sigma}g_{\mu\sigma,\lambda}}{4} + \frac{g^{44}g_{44,\lambda}g^{\lambda\beta}g_{44,\beta}}{4}$$

$$= -\varepsilon\Phi\left(g^{\lambda\beta}{}_{,\lambda}\Phi_\beta + g^{\lambda\beta}\Phi_{\beta,\lambda} + \frac{g^{\lambda\beta}g^{\mu\sigma}g_{\mu\sigma,\lambda}\Phi_\beta}{2}\right)$$

$$= -\varepsilon\Phi g^{\mu\nu}\Phi_{\mu;\nu}\ . \tag{2.46}$$

Here we have obtained the last line by noting that $\Phi_{\beta,\lambda} = \Phi_{\beta,\lambda} - \Gamma^\sigma_{\beta\lambda}\Phi_\sigma$ implies

$$g^{\beta\lambda}\Phi_{\beta,\lambda} + \frac{g^{\beta\lambda}g^{\sigma\mu}g_{\beta\lambda,\mu}\Phi_\sigma}{2} = g^{\mu\nu}\Phi_{\mu;\nu} + \frac{g^{\beta\lambda}g^{\sigma\mu}g_{\mu\beta,\lambda}\Phi_\sigma}{2} + \frac{g^{\beta\lambda}g^{\sigma\mu}g_{\mu\lambda,\beta}\Phi_\sigma}{2}\ , \tag{2.47}$$

and that $\left(\delta^\mu_\nu\right)_{,\mu} = 0$ implies $\left(g^{\lambda\sigma}{}_{,\lambda} + g^{\beta\lambda}g^{\sigma\mu}g_{\mu\beta,\lambda}\right)\Phi_\sigma = 0$. Putting (2.46) in (2.45) gives lastly

$$R_{44} = -\varepsilon\Phi\Box\Phi - \frac{\overset{*}{g}^{\lambda\beta}\overset{*}{g}_{\lambda\beta}}{2} - \frac{g^{\lambda\beta}\overset{**}{g}_{\lambda\beta}}{2} + \frac{\overset{*}{\Phi}g^{\lambda\beta}\overset{*}{g}_{\lambda\beta}}{2\Phi} - \frac{g^{\mu\beta}g^{\lambda\sigma}\overset{*}{g}_{\lambda\beta}\overset{*}{g}_{\mu\sigma}}{4}\ , \tag{2.48}$$

where $\Box\Phi \equiv g^{\mu\nu}\Phi_{\mu;\nu}$ defines the 4D curved-space box operator. Equations (2.43) and (2.48) can be used with the 5D field equations (2.1) which we repeat here:

$$R_{AB} = 0 \ . \tag{2.49}$$

Then $\hat{R}_{\alpha\beta} = 0$ in (2.43) gives

$$R_{\alpha\beta} = \frac{\Phi_{\alpha;\beta}}{\Phi} - \frac{\varepsilon}{2\Phi^2}\left(\frac{\overset{*}{\Phi}\overset{*}{g}_{\alpha\beta}}{\Phi} - \overset{**}{g}_{\alpha\beta} + g^{\lambda\mu}\overset{*}{g}_{\alpha\lambda}\overset{*}{g}_{\beta\mu} - \frac{g^{\mu\nu}\overset{*}{g}_{\mu\nu}\overset{*}{g}_{\alpha\beta}}{2}\right). \tag{2.50}$$

And $R_{44} = 0$ in (2.48) gives

$$\varepsilon\Phi\Box\Phi = -\frac{g^{\lambda\beta}\overset{*}{g}_{\lambda\beta}}{4} - \frac{g^{\lambda\beta}\overset{**}{g}_{\lambda\beta}}{2} + \frac{\overset{*}{\Phi}g^{\lambda\beta}\overset{*}{g}_{\lambda\beta}}{2\Phi} \ , \tag{2.51}$$

where we have noted that $\left(\delta^{\mu}_{\nu}\right)_{,4} = 0$ implies $g^{\mu\beta}g^{\lambda\sigma}\overset{*}{g}_{\lambda\beta}\overset{*}{g}_{\mu\sigma} + g^{\mu\sigma}\overset{*}{g}_{\mu\sigma} = 0$. From (2.50) we can form the 4D Ricci curvature scalar $R = g^{\alpha\beta}R_{\alpha\beta}$. Eliminating the covariant derivative using (2.51), and again using $\left(\delta^{\mu}_{\nu}\right)_{,4} = 0$ to eliminate some terms, gives

$$R = \frac{\varepsilon}{4\Phi^2}\left[g^{\mu\nu}\overset{*}{g}_{\mu\nu} + \left(g^{\mu\nu}\overset{*}{g}_{\mu\nu}\right)^2\right] \ . \tag{2.52}$$

With (2.50) and (2.52), we are now in a position to define if we wish an energy-momentum tensor in 4D via $8\pi T_{\alpha\beta} \equiv R_{\alpha\beta} - Rg_{\alpha\beta}/2$. It is

$$8\pi T_{\alpha\beta} = \frac{\Phi_{\alpha;\beta}}{\Phi} - \frac{\varepsilon}{2\Phi^2}\left\{\frac{\overset{*}{\Phi}\overset{*}{g}_{\alpha\beta}}{\Phi} - \overset{**}{g}_{\alpha\beta} + g^{\lambda\mu}\overset{*}{g}_{\alpha\lambda}\overset{*}{g}_{\beta\mu} - \frac{g^{\mu\nu}\overset{*}{g}_{\mu\nu}\overset{*}{g}_{\alpha\beta}}{2} + \frac{g_{\alpha\beta}}{4}\left[g^{\mu\nu}\overset{*}{g}_{\mu\nu} + \left(g^{\mu\nu}\overset{*}{g}_{\mu\nu}\right)^2\right]\right\} \tag{2.53}$$

.

Provided we use this energy-momentum tensor, Einstein's 4D field equations (2.3) or $G_{\alpha\beta} = 8\pi T_{\alpha\beta}$ will of course be satisfied.

The mathematical expression (2.53) has good properties. It is a symmetric tensor that has a part which depends on derivatives of Φ with respect to the usual coordinates x^{0123}, and a part which depends on derivatives of other metric coefficients with respect to the extra coordinate x^4. [The first term in (2.53) is implicitly symmetric because it depends on the second partial derivative, while the other terms are explicitly symmetric.] It is also compatible with what is known about the recovery of 4D properties of matter from apparently empty 5D solutions of Kaluza-Klein theory. Thus the cosmological case studied in Section 2.3 agrees with (2.53) and has matter which owes its characteristics largely to the x^4-dependency of $g_{\alpha\beta}$ in that relation. While the soliton case studied in Section 2.4 agrees with (2.53) and has matter which depends on the first or scalar term in that relation. With (2.53) and preceding relations, the case where there is no dependency on x^4 becomes transparent. Then (2.51) becomes the scalar wave equation for the extra part of the metric ($g^{\mu\nu}\Phi_{\mu;\nu} = 0$ with $g_{44} = \varepsilon\Phi^2$). And (2.53) gives $T = T_{\alpha\beta}g^{\alpha\beta} = 0$, which implies a radiation-like equation of state. However, in general there must be x^4-dependence if we are to recover more complex equations of state from solutions of $R_{AB} = 0$.

These field equations have 4 other components we have not so far considered, namely $R_{4\alpha} = 0$. This relation by (2.35) expanded is

$$R_{4\alpha} = \left(\Gamma^{\lambda}_{4\alpha}\right)_{,\lambda} + \left(\Gamma^{4}_{4\alpha}\right)_{,4} - \left(\Gamma^{\lambda}_{4\lambda}\right)_{,\alpha} - \left(\Gamma^{4}_{44}\right)_{,\alpha} + \Gamma^{\lambda}_{4\alpha}\Gamma^{A}_{\lambda A} + \Gamma^{4}_{4\alpha}\Gamma^{A}_{4A} - \Gamma^{A}_{4\mu}\Gamma^{\mu}_{\alpha A} - \Gamma^{D}_{44}\Gamma^{4}_{\alpha D} \quad . \tag{2.54}$$

The Christoffel symbols here are tabulated in (2.40) and cause (2.54) to become

$$R_{4\alpha} = \frac{g^{44}g^{\lambda\beta}}{4}\left(\dot{g}_{\lambda\beta}\, g_{44,\alpha} - g_{44,\beta}\,\dot{g}_{\alpha\lambda}\right) + \frac{g^{\lambda\mu}{}_{,\lambda}\,\dot{g}_{\mu\alpha}}{2}$$

$$+ \frac{g^{\lambda\mu}\,\dot{g}_{\mu\alpha,\lambda}}{2} - \frac{g^{\lambda\beta}{}_{,\alpha}\,\dot{g}_{\lambda\beta}}{2} - \frac{g^{\lambda\beta}\,\dot{g}_{\lambda\beta,\alpha}}{2} + \frac{g^{\lambda\sigma}g^{\mu\beta}\,\dot{g}_{\sigma\alpha}\,g_{\mu\beta,\lambda}}{4} + \frac{g^{\mu\beta}\,\dot{g}_{\mu\beta,\alpha}}{4} \quad . \tag{2.55}$$

Here we have done some algebra using $\left(g_{44}g^{44}\right)_{,\alpha \text{ and } 4}=0$ or $\overset{*}{g}{}^{44}\,g_{44,\alpha}-g^{44}{}_{,\alpha}\,\overset{*}{g}_{44}=0$, and $\left(\delta^{\beta}_{\lambda}\right)_{,4}=0$ or $g^{\lambda\nu}g_{\mu\nu}g^{\mu\sigma}+\overset{*}{g}{}^{\lambda\sigma}=0$. [We also note in passing that one can use $\left(\Gamma^{4}_{4\alpha}\right)_{,4}=\left(\Gamma^{4}_{44}\right)_{,\alpha}$ in (2.54) and obtain an alternative form of (2.55) with the last term replaced by $g^{\mu\beta}{}_{,\alpha}\,\overset{*}{g}_{\mu\beta}/4$.] While (2.55) may be useful in other computations, it is helpful for our purpose here to rewrite it as

$$R_{4\alpha}=\frac{\partial}{\partial x^{\beta}}\left(\frac{g^{\beta\lambda}\overset{*}{g}_{\lambda\alpha}}{2}\right)-\frac{\partial}{\partial x^{\alpha}}\left(\frac{g^{\mu\nu}\overset{*}{g}_{\mu\nu}}{2}\right)+\left(\frac{g^{\mu\beta}g_{\mu\beta,\lambda}}{2}\right)\left(\frac{g^{\lambda\sigma}\overset{*}{g}_{\sigma\alpha}}{2}\right)-\left(\frac{g^{\lambda\beta}g_{\beta\mu,\alpha}}{2}\right)\left(\frac{g^{\mu\sigma}\overset{*}{g}_{\sigma\lambda}}{2}\right)$$

$$-\frac{g^{44}g_{44,\beta}}{2}\left(\frac{g^{\beta\lambda}\overset{*}{g}_{\lambda\alpha}}{2}-\frac{\delta^{\beta}_{\alpha}g^{\mu\nu}\overset{*}{g}_{\mu\nu}}{2}\right). \tag{2.56}$$

Noting that $\partial/\partial x^{\alpha}=\delta^{\beta}_{\alpha}\left(\partial/\partial x^{\beta}\right)$ and that $-g^{44}g_{44,\beta}/2=\sqrt{g_{44}}\left(\partial/\partial x^{\beta}\right)\left(1/\sqrt{g_{44}}\right)$ allows us to obtain finally

$$\frac{R_{4\alpha}}{\sqrt{g_{44}}}=\frac{\partial}{\partial x^{\beta}}\left[\frac{1}{2\sqrt{g_{44}}}\left(g^{\beta\lambda}\overset{*}{g}_{\lambda\alpha}-\delta^{\beta}_{\alpha}g^{\mu\nu}\overset{*}{g}_{\mu\nu}\right)\right]$$

$$+\left(\frac{g^{\mu\beta}g_{\mu\beta,\lambda}}{2}\right)\left(\frac{g^{\lambda\sigma}\overset{*}{g}_{\sigma\alpha}}{2\sqrt{g_{44}}}\right)-\left(\frac{g^{\lambda\beta}g_{\beta\mu,\alpha}}{2}\right)\left(\frac{g^{\mu\sigma}\overset{*}{g}_{\sigma\lambda}}{2\sqrt{g_{44}}}\right). \tag{2.57}$$

This form suggests we should introduce the 4-tensor

$$P^{\beta}_{\alpha}\equiv\frac{1}{2\sqrt{g_{44}}}\left(g^{\beta\lambda}\overset{*}{g}_{\lambda\alpha}-\delta^{\beta}_{\alpha}g^{\mu\nu}\overset{*}{g}_{\mu\nu}\right). \tag{2.58}$$

The divergence of this is

$$P^{\beta}_{\alpha;\beta}=\left(P^{\beta}_{\alpha}\right)_{,\beta}+\Gamma^{\beta}_{\beta\mu}P^{\mu}_{\alpha}-\Gamma^{\mu}_{\alpha\beta}P^{\beta}_{\mu}\quad,$$

which when written out in full may be shown to be the same as the r.h.s. of (2.57). The latter therefore reads

$$\frac{R_{4\alpha}}{\sqrt{g_{44}}} = P^{\beta}_{\alpha;\beta} \quad . \tag{2.59}$$

The field equations (2.49) as $R_{4\alpha} = 0$ can then be summed up by the relations

$$P^{\beta}_{\alpha;\beta} = 0 \ ,$$

$$P^{\beta}_{\alpha} \equiv \frac{1}{2\sqrt{g_{44}}}\left(g^{\beta\sigma} \overset{*}{g}_{\sigma\alpha} - \delta^{\beta}_{\alpha} g^{\mu\nu} \overset{*}{g}_{\mu\nu} \right) . \tag{2.60}$$

These have the appearance of conservation laws for P^{β}_{α}. The fully covariant form and associated scalar for the latter are:

$$P_{\alpha\beta} = \frac{1}{2\sqrt{g_{44}}}\left(\overset{*}{g}_{\alpha\beta} - g_{\alpha\beta} g^{\mu\nu} \overset{*}{g}_{\mu\nu} \right)$$

$$P = \frac{-3 g^{\lambda\sigma} \overset{*}{g}_{\lambda\sigma}}{2\sqrt{g_{44}}} \quad . \tag{2.61}$$

We will examine these quantities elsewhere, but here we comment that while our starting gauge (2.34) removed the explicit electromagnetic potentials, the field equations $R_{4\alpha} = 0$ or (2.60) are of electromagnetic type.

It is apparent from the working in this section that the starting conditions (2.34) provide a convenient way to split the 5D field equations $R_{AB} = 0$ into 3 sets: The 5D equations $\hat{R}_{\alpha\beta} = 0$ give a set of equations in the 4D Ricci tensor $R_{\alpha\beta}$ (2.50); the 5D equation $R_{44} = 0$ gives a wave-like equation in the scalar potential (2.51); and the 5D equations $R_{4\alpha} = 0$ can be expressed as a set of 4D conservation laws (2.60). Along the way we also obtain some other useful relations, notably an expression for the 4D Ricci scalar in terms of the dependency of the 4D metric on the

extra coordinate (2.52). However, the physically most relevant expression is an effective or induced 4D energy-momentum tensor (2.53). Another way to express these results is to say that the 15 field equations $R_{AB} = 0$ of (2.1) or $G_{AB} = 0$ of (2.2) can always be split into 3 sets which make physical sense provided the metric is allowed to depend on the extra coordinate x^4. These sets consist of 4 conservation equations of electromagnetic type, 1 equation for the scalar field of wave type, and 10 equations for fields and matter of gravitational type. In fact, the last are Einstein's equations (2.3) of general relativity, with matter induced from the extra dimension.

2.6 Conclusion

The idea of embedding $G_{\alpha\beta} = 8\pi T_{\alpha\beta}$ (4D) in $R_{AB} = 0$ (5D) is motivated by the wish to explain classical properties of matter rather than merely accepting them as given. In application to the cosmological case it works straightforwardly, and gives back 5D geometric quantities which are identical to the 4D density and pressure (Section 2.3). This is important: what we derive from the 5D equations is not something esoteric but ordinary matter. In application to the soliton or 1-body case, the idea leads to a class of radiation-like solutions which contains as a very special case the Schwarzschild solution (Section 2.4). In general application to neutral matter, the properties of the latter turn out to be intimately connected to x^4-dependency of the metric (Section 2.5). Induced-matter theory actually admits a wide variety of equations of state (Ponce de Leon and Wesson 1993). But in the matter gauge at least, independence from x^4 implies radiation-like matter, while dependence on x^4 implies other kinds of matter.

The theoretical basis we have demonstrated in this chapter leads naturally to the question of observations, particularly with regard to the solitons. As mentioned above, there is a class of these in 5D rather than the unique Schwarzschild solution of 4D, because Birkhoff's theorem in

its conventional form does not apply. Indeed, there are known exact solutions which represent time-dependent solitons (Liu, Wesson and Ponce de Leon 1993; Wesson, Liu and Lim 1993). And there is known an exact solution which is x^4-dependent and Schwarzschild-like (Mashhoon, Liu and Wesson, 1994). We will return to the latter, where we will find that it implies the same dynamics as in general relativity and so poses no problem. However, there remains the question of the observational status of the standard solitons. This has been investigated by a number of people, most of whom were not working in the induced-matter picture (see Overduin and Wesson 1997). Here, we can regard the soliton as a concentration of matter at the centre of ordinary space, and ask about the motions of test particles at large distances. Specifically, we ask what constraints we can put on the soliton 1-body metric from the classical tests of relativity.

References (Chapter 2)

Abolghasem, G., Coley, A.A., McManus, D.J. 1996, J. Math. Phys. 37, 361.

Billyard, A., Wesson, P.S., Kalligas, D. 1995, Int. J. Mod. Phys. D 4, 639.

Billyard, A., Wesson, P.S. 1996, Gen. Rel. Grav. 28, 129.

Brout, R., Englert, F., Gunzig, E. 1978, Ann. Phys. (New York) 115, 78.

Campbell, J.E. 1926, A Course of Differential Geometry (Clarendon, Oxford).

Chatterjee, S. 1990, Astron. Astrophys. 230, 1.

Chatterjee, S., Sil, A. 1993, Gen. Rel. Grav. 25, 307.

Chatterjee, S., Panigrahi, D., Banerjee, A. 1994, Class. Quant. Grav. 11, 371.

Davidson, A., Owen, D. 1985, Phys. Lett. B 155, 247.

Eisenhart, L.P. 1949, Riemannian Geometry (Princeton U. Press., Princeton).

Gross, D.J., Perry, M.J. 1983, Nucl. Phys. B 226, 29.

Guth, A.H. 1981, Phys. Rev. D 23, 347.

Kramer, D., Stephani, H., MacCallum, M., Herlt, E. 1980, Exact Solutions of Einstein's Field Equations (Cambridge U. Press, Cambridge).

Lidsey, J.E., Romero, C., Tavakol, R., Rippl, S. 1997, Class. Quant. Grav. 14, 865.

Liu, H., Wesson, P.S., Ponce de Leon, J. 1993, J. Math. Phys. 34, 4070.

Liu, H., Wesson, P.S. 1994, Int. J. Mod. Phys. D 3, 627.

Liu, H., Mashhoon, B. 1995, Ann. Phys. (Leipzig) 4, 565.

Mashhoon, B., Liu, H., Wesson, P.S. 1994, Phys. Lett. B 331, 305.

Overduin, J., Wesson, P.S. 1997, Phys. Rep. 283, 303.

Ponce de Leon, J. 1988, Gen. Rel. Grav. 20, 539.

Ponce de Leon, J., Wesson, P.S. 1993, J. Math. Phys. 34, 4080.

Rippl, S., Romero, C., Tavakol, R. 1995, Class. Quant. Grav. 12, 2411.

Romero, C., Tavakol. R., Zalaletdinov, R. 1996, Gen. Rel. Grav. 28, 365.

Schouten, J.A., Struik, D.J. 1921, Am. J. Math. 43, 213.

Sorkin, R.D. 1983, Phys. Rev. Lett. 51, 87.

Tangherlini, F.R. 1963, Nuovo Cim. 27 (Ser. 10), 636.

Wesson, P.S. 1992a, Astrophys. J. 394, 19.

Wesson, P.S. 1992b, Phys. Lett. B 276, 299.

Wesson, P.S., Ponce de Leon, J. 1992, J. Math. Phys. 33, 3883.

Wesson, P.S., Liu, H., Lim, P. 1993, Phys. Lett. B 298, 69.

Wesson, P.S. 1994, Astrophys. J. 436, 547.

Wesson, P.S., Ponce de Leon, J. 1994, Class. Quant. Grav. 11, 1341.

3. THE CLASSICAL AND OTHER TESTS IN 5D

"You need to be constrained" (Clifford Will, St. Louis, 1992)

3.1 Introduction

Irrespective of which interpretation one makes of 5D general relativity, the observational status of the soliton or 1-body solutions is central. The classical tests of relativity involve the 4D Schwarzschild solution applied to the solar system. These include the perihelion advance of Mercury, the gravitational redshift effect, and the deflection of light by the Sun. To these were added the time-delay (or radar-ranging) test for electromagnetic radiation passing near the Sun, and more recently the change due to gravitational radiation of the orbital dynamics of the binary pulsar system. Einstein's 4D theory agrees with data on these things, in some cases to an accuracy of order 0.1% (Adler, Bazin and Schiffer 1975; Will 1992). Kaluza-Klein 5D theory can obviously be constrained by these data. However, it also leads to new tests involving the geodetic precession of a gyroscope and possible departures from the equivalence principle, both of which can be tested by spacecraft experiments.

In this chapter, we treat the 5D dynamics of the class of soliton solutions examined in the preceding chapter. We proceed by first putting the 5D metric into a form that resembles the 4D one. Then because it is the physically most interesting case, we look at the orbits of photons. After, we treat the orbits of massive particles and the perihelion advance problem, followed by the gravitational redshift effect. There follow two sections that deal, respectively, with the precession of a spinning object in 4D and 5D, and the status of the 4D equivalence principle versus 5D geodesic motion.

3.2 The 1-Body Metric

In 4D, the Schwarzschild solution is given in its usual and (3D) isotropic forms by

$$ds^2 = \left(1-\frac{2M}{r}\right)dt^2 - \left(1-\frac{2M}{r}\right)^{-1} dr^2 - r^2\, d\Omega^2$$

$$ds^2 = \left(\frac{1-M/2\rho}{1+M/2\rho}\right)^2 dt^2 - \left(1+\frac{M}{2\rho}\right)^4 \left(d\rho^2 + \rho^2 d\Omega^2\right). \qquad (3.1)$$

Here as before, we are absorbing the speed of light c and the gravitational constant G via a choice of units which renders c = 1, G = 1; and we are using coordinates $x^0 = t$, $x^{123} = r\theta\phi$ $\left(d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\phi^2\right)$. The 2 forms of (3.1) are related by the well-known coordinate transformation

$$\rho = \frac{1}{2}\left[r - M + \sqrt{r(r-2M)}\right]. \qquad (3.2)$$

The Schwarzschild solution describes the gravitational field outside a mass M, and up to coordinate transformations like (3.2) is unique in 4D by virtue of Birkhoff's theorem.

In 5D, the 1-body problem is described by a class of solutions, most frequently described in the forms due to Gross and Perry (1983) and Davidson and Owen (1985). In the preceding chapter, it was convenient to discuss induced matter by using the terminology of the latter authors; but in this chapter, it will be convenient to discuss dynamics by using the terminology of the former authors. Thus the line element in isotropic coordinates is

$$dS^2 = \left(\frac{1-M/2\rho}{1+M/2\rho}\right)^{2/\alpha} dt^2 - \left(\frac{1-M/2\rho}{1+M/2\rho}\right)^{2(\alpha-\beta-1)/\alpha} \left(1+\frac{M}{2\rho}\right)^4 \left[d\rho^2 + \rho^2 d\Omega^2\right]$$

$$- \left(\frac{1-M/2\rho}{1+M/2\rho}\right)^{2\beta/\alpha} dl^2 . \qquad (3.3)$$

Here $x^4 = l$ is the new coordinate, and there are 2 dimensionless constants α, β which are related by the consistency relation $\alpha^2 = \beta^2 + \beta + 1$ (see Chapter 2). Thus we have a 2-parameter class, where M is regarded as fixed by observations and α, β can be considered as having a free

range. (However, for induced-matter theory as discussed in Section 2.4, the sizes of α and β are related to the density and pressure of the 4D matter associated with the 5D solution.) To cause the 5D solutions (3.3) to resemble the 4D Schwarzschild solution in its usual form we can use (3.2). It is convenient also to redefine the constants via $a \equiv 1/\alpha$, $b \equiv \beta/\alpha$ and introduce $A(r) \equiv 1-2M/r$. Then (3.3) becomes

$$dS^2 = A^a dt^2 - A^{-(a+b)} dr^2 - A^{1-a-b} r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) - A^b dl^2 \,, \tag{3.4}$$

and the aforementioned consistency relation becomes $1 = a^2 + ab + b^2$. Incidentally, this is invariant under a change in signs of *a, b* and a swap of *a, b*. The significance of this is obscure. The choice $a = 1$, $b = 0$ means that the 4D part of (3.4) becomes identical to the 4D Schwarzschild solution (3.1), which is therefore embedded.

We will require below the Christoffel symbols associated with (3.4). The nonzero ones are:

$$\Gamma^0_{10} = \frac{aM}{r(r-2M)} \qquad \Gamma^1_{00} = \frac{aM}{r^2}\left(\frac{1-2M}{r}\right)^{2a+b-1}$$

$$\Gamma^1_{11} = \frac{-(a+b)M}{r(r-2M)} \qquad \Gamma^1_{22} = -(1-a-b)M - (r-2M)$$

$$\Gamma^1_{33} = -\left[(1-a-b)M + r - 2M\right]\sin^2\theta \qquad \Gamma^1_{44} = -\frac{bM}{r^2}\left(\frac{1-2M}{r}\right)^{a+2b-1}$$

$$\Gamma^2_{12} = (1-a-b)\frac{M}{r(r-2M)} + \frac{1}{r} \qquad \Gamma^2_{33} = -\sin\theta\cos\theta$$

$$\Gamma^3_{13} = (1-a-b)\frac{M}{r(r-2M)} + \frac{1}{r} \qquad \Gamma^3_{23} = \cot\theta$$

$$\Gamma^4_{14} = \frac{bM}{r(r-2M)} \,. \tag{3.5}$$

The Lagrangian associated with (3.4) is

$$L = A^a \dot{t}^2 - A^{-(a+b)} \dot{r}^2 - A^{1-a-b} r^2 \left(\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2 \right) - A^b \dot{l}^2 \ , \tag{3.6}$$

from which equations of motion can be derived by the usual variational principle. In the above, a dot denotes the total derivative w.r.t. S for a massive particle or the total derivative w.r.t. some affine parameter λ for a photon. Both cases have been considered in a comprehensive account of dynamical tests of 5D gravity by Kalligas, Wesson and Everitt (1995). The light-deflection problem has been analyzed using a somewhat different method but with the same result by Lim, Overduin and Wesson (1995). We use these accounts to look first at the case of null geodesics.

3.3 Photon Orbits

For these, $L = 0$ in (3.6) and without loss of generality we put $\theta = \pi/2$, $\dot{\theta} = 0$. The deflection of light can be worked out by noting from (3.6) that there are 3 constants of the motion associated with the t, ϕ, l directions:

$$j \equiv A^a \dot{t} \qquad h \equiv A^{(1-a-b)} r^2 \dot{\phi} \qquad k \equiv A^b \dot{l} \ . \tag{3.7}$$

The line element vanishes for null geodesics, $dS^2 = 0$. This into equation (3.4) yields, after dividing by $\dot{\phi}^2$,

$$A^a \frac{\dot{t}^2}{\dot{\phi}^2} - A^{-(a+b)} \frac{\dot{r}^2}{\dot{\phi}^2} - A^{1-a-b} r^2 - A^b \frac{\dot{l}^2}{\dot{\phi}^2} = 0 \ . \tag{3.8}$$

By making use of the constants of the motion (3.7) we can rewrite equation (3.8) in a form containing only $\dot{r}^2 / \dot{\phi}^2 = \left(dr / d\phi \right)^2$. We obtain

$$\left(\frac{dr}{d\phi} \right)^2 - \left(A^{2-2a-b} j^2 - A^{2-a-2b} k^2 \right) \frac{r^4}{h^2} + A r^2 = 0 \ . \tag{3.9}$$

Rewriting in terms of $u \equiv 1/r$, this gives

$$\left(\frac{du}{d\phi} \right)^2 - \left(A^{2-2a-b} j^2 - A^{2-a-2b} k^2 \right) \frac{1}{h^2} + A u^2 = 0 \ . \tag{3.10}$$

This in the weak-field limit $Mu = M/r << 1$ gives

$$\left(\frac{du}{d\phi}\right)^2 + u^2 = f + \varepsilon p u + \varepsilon u^3 . \tag{3.11}$$

Here we have written $\varepsilon = 2M$ and

$$f \equiv \frac{j^2}{h^2} - \frac{k^2}{h^2}$$

$$p \equiv (2-a-2b)\frac{k^2}{h^2} - (2-2a-b)\frac{j^2}{h^2} \tag{3.12}$$

are constants introduced for future convenience.

We are interested in the deviation from a straight path for a photon, coming from plus infinity and escaping to minus infinity, due to the presence of a central body. At the point of closest approach u_0 we have $du/d\phi = 0$. This into equation (3.11) yields

$$f = u_0^2(1 - \varepsilon u_0 - \varepsilon p/u_0). \tag{3.13}$$

By inserting this back into (3.11) we obtain

$$\left(\frac{du}{d\phi}\right)^2 + u^2 = u_0^2\left[1 - \varepsilon u_0 - (\varepsilon p/u_0)\right] + \varepsilon p u + \varepsilon u^3 . \tag{3.14}$$

As we consider the weak-field regime, ε is a small quantity. Working to first order, we find for u the expression

$$u(\phi) = u_0 \sin\phi + \frac{\varepsilon u_0^2}{2}(1 + \cos^2\phi - \sin\phi) + \frac{\varepsilon p}{2}(1 - \sin\phi) + \varepsilon\left(u_0^2 + \frac{p}{2}\right)\cos\phi , \tag{3.15}$$

or equivalently

$$u(\phi) = \left[u_0\left(1 - \frac{\varepsilon u_0}{2}\right) - \frac{\varepsilon p}{2}\right]\sin\phi + \frac{\varepsilon u_0^2}{2}(1-\cos\phi)^2 + \frac{\varepsilon p}{2}(1-\cos\phi) . \tag{3.16}$$

In the limit where ε goes to zero, this gives $u(\phi) = u_0 \sin\phi$, which is a straight line for the path of a photon originating from $+\infty$ at $\phi = 0$ and escaping to $-\infty$ along the direction $\phi = \pi$. For

nonzero ε the photon will deflect and escape to -∞ along the direction $\phi = \pi + \delta$. In the weak-field limit we expect δ to be small, so we will only keep terms linear in ε and δ in the following. We can then find δ with the help of (3.16), by demanding that $u(\phi = \pi + \delta) = 0$. We obtain

$$\delta = \frac{4M}{r_0} + 2Mpr_0 \ . \tag{3.17}$$

The first term here is the familiar result of 4D general relativity.

The deviation from this result dictated by the second term has a somewhat unusual form, as it might appear to imply that the deflection angle would grow with the impact parameter r_0. However, as p involves the 'angular momentum' constant h in the denominator it will be proportional to $1/r_0^2$, so $2Mpr_0$ will also diminish with increasing r_0. It can be seen from (3.12) and (3.17) that $2Mpr_0$ does not vanish even if the photon has no velocity component in the fifth dimension, unless we also have $a = 1$, $b = 0$ which is the Schwarzschild case. To demonstrate this we insert $k = 0$ into (3.12). Then using (3.7) and (3.17) we can write the deflection angle as

$$\delta = \frac{(4a + 2b)M}{r_0} \ , \tag{3.18}$$

which clearly reproduces the usual 4D result when $a = 1$, $b = 0$. As an illustration, let us consider a photon passing the Sun with impact parameter $r_0 = R_\odot$, and assume $b = 0.0075$. Then the consistency relation gives $a = 0.9962$. For the case k = 0, the values of the second and first terms in (3.17) are found to be in the ratio 0.004, which represents a small decrease in the deflection angle. In terms of the reported accuracy for this type of experiment (Will 1992), such a departure would be just detectable if the Sun had the noted value of b. Of course, while b is a constant it is not a universal one, so while it may be small for the Sun it could be more significant in other systems. In this regard, it should be noted that (3.17) and (3.18) allow not

only for light attraction but also for light repulsion for certain values of the parameters. This demonstrates the peculiar consequences that the presence of an extra dimension may lead to as concerns light deflection.

The time delay of photons passing through a curved spacetime, or radar ranging, is often referred to as the fourth classical test of general relativity (see Will 1992). We wish to calculate the time delay for the metric (3.4), assuming that the latter describes the field outside the Sun. To do this, we imagine that an observer on Earth emits radar signals which pass by the Sun and are reflected back by another planet, where all 3 objects lie in the plane $\theta = \pi/2$. Then to a first approximation we can use the $\varepsilon \to 0$ expression noted above for the path, namely $u(\phi) = u_0 \sin\phi$ (the actual curvature of the path makes a negligible contribution to the time delay). Differentiating this and reintroducing radii instead of angles we obtain

$$r^2 d\phi^2 = \frac{r_0^2}{\left(r^2 - r_0^2\right)} dr^2 \quad . \tag{3.19}$$

If we now multiply (3.8) by $\dot{\phi}^2 / \dot{t}^2$ and use (3.19) and (3.7) we find

$$A^a = A^{-(a+b)}\left(\frac{dr}{dt}\right)^2 + A^{1-a-b}\frac{r_0^2}{\left(r^2 - r_0^2\right)}\left(\frac{dr}{dt}\right)^2 + A^{2a-b}\left(\frac{k}{j}\right)^2, \tag{3.20}$$

which can be solved for

$$\frac{dr}{dt} = \left\{\left[1 - A^{a-b}\left(\frac{k}{j}\right)^2\right] \Big/ \left[A^{-(2a+b)} + A^{1-2a-b}\frac{r_0^2}{\left(r^2 - r_0^2\right)}\right]\right\}^{1/2} \quad . \tag{3.21}$$

The coordinate time for the photon's round trip can then be found by summing contributions of the form:

$$\Delta t = \int \left\{\left[A^{-(2a+b)} + A^{1-2a-b}\frac{r_0^2}{\left(r^2 - r_0^2\right)}\right] \Big/ \left[1 - A^{a-b}\left(\frac{k}{j}\right)^2\right]\right\}^{1/2} dr \quad . \tag{3.22}$$

In the weak-field regime the integrand here can be written in the form

$$\frac{r}{\sqrt{r^2-r_0^2}}\left\{1+\frac{1}{2}\left(\frac{k}{j}\right)^2+\left[(2a+b)+\frac{3b}{2}\left(\frac{k}{j}\right)^2\right]\frac{M}{r}-\left[1+\left(\frac{k}{j}\right)^2\right]\frac{Mr_0^2}{r^3}\right\}. \tag{3.23}$$

This back into (3.22) gives, after summing, a round trip time of

$$\Delta t_{round\ trip}=2\left\{\left[1+\frac{1}{2}\left(\frac{k}{j}\right)^2\right]\left(\sqrt{r_p^2-r_0^2}+\sqrt{r_E^2-r_0^2}\right)+M\left[(2a+b)+\frac{3b}{2}\left(\frac{k}{j}\right)^2\right]\right.$$

$$\left.\times\left[\ln\left(\frac{r_p+\sqrt{r_p^2-r_0^2}}{r_0}\right)+\ln\left(\frac{r_E+\sqrt{r_E^2-r_0^2}}{r_0}\right)\right]-M\left[1+\frac{1}{2}\left(\frac{k}{j}\right)^2\right]\left[\frac{\sqrt{r_p^2-r_0^2}}{r_p}+\frac{\sqrt{r_E^2-r_0^2}}{r_E}\right]\right\}. \tag{3.24}$$

Here r_E, r_p are the radius measures of the Earth (emitter) and planet (reflector), and r_0 is the impact parameter as above. The coordinate time delay (3.24) has an associated proper time delay

$$\Delta\tau_{round\ trip}=\left(1-\frac{2M}{r}\right)^a\Delta t_{round\ trip}\equiv\left(1-\frac{2aM}{r}\right)\Delta t_{round\ trip}\quad . \tag{3.25}$$

Both (3.24) and (3.25) give back the result of 4D general relativity for $a = 1$, $b = 0$, $k = 0$. Any deviation from these values in our solar system must be such that the resulting fractional change in the time delay be less than about 0.1% (Will 1992). For $k = 0$, the prefactor of the logarithmic terms in (3.24) becomes negative or null for the same values of a and b as in the light-deflection problem. Thus there is again the possibility of peculiar effects due to an extra dimension in other astronomical systems.

To finish this section, we wish to comment on another difference between 4D and 5D which concerns circular photon orbits. In the 4D Schwarzschild metric, photons can be set into circular motion about a central object of mass M for a particular value of the radial coordinate, namely $r = 3M$. To find out what the corresponding value is for the 5D metric (3.4), we can take the Lagrangian (3.6) and write the Lagrange equation for the radial direction as

$$-2A^{-(a+b)}\ddot{r}+2(a+b)A^{-(a+b+1)}A_{,r}\dot{r}^2-aA^{a-1}A_{,r}\dot{t}^2-(a+b)A^{-(a+b+1)}A_{,r}\dot{r}^2$$

$$+(1-a-b)A^{-(a+b)}A_{,r}r^2\dot{\phi}^2+2A^{1-a-b}r\dot{\phi}^2+bA^{b-1}A_{,r}\dot{l}^2=0\ . \tag{3.26}$$

For circular orbits r is constant, so this simplifies to

$$-aA^{a-1}A_{,r}\dot{t}^2+(1-a-b)A^{-(a+b)}A_{,r}r^2\dot{\phi}^2+2A^{1-a-b}r\dot{\phi}^2+bA^{b-1}A_{,r}\dot{l}^2=0\ . \tag{3.27}$$

If we divide this equation by $\dot{\phi}^2$ and use $\dot{l}^2/\dot{\phi}^2=k^2A^{2-2a-4b}r^4/h^2$ from (3.6), we obtain

$$\frac{\dot{t}^2}{\dot{\phi}^2}=\frac{(1-a-b)}{a}A^{1-2a-b}r^2+A^{2-2a-b}\frac{r^3}{Ma}+A^{2-3a-3b}\frac{b^2k^2r^4}{ah^2}\ . \tag{3.28}$$

But from (3.8) with $\dot{r}=0$ we find another expression for $\dot{t}^2/\dot{\phi}^2$, namely

$$\frac{\dot{t}^2}{\dot{\phi}^2}=A^{1-2a-b}r^2+\frac{k^2}{h^2}A^{2-3a-3b}r^4\ . \tag{3.29}$$

For consistency, (3.28) and (3.29) must match, so

$$(1-2a-b)r^2+\frac{A}{M}r^3+(b-a)A^{1-a-2b}\frac{k^2r^4}{h^2}=0\ . \tag{3.30}$$

The solutions to this equation give the permitted radii for the orbits. For the case of negligible motion along the fifth dimension $(k\to 0)$ we obtain

$$r=(1+2a+b)M\ . \tag{3.31}$$

This reduces back to $r = 3M$ for the Schwarzschild case with $a = 1$, $b = 0$. However, it is clear that the sizes of circular photon orbits can in principle be quite different for other cases.

3.4 Particle Orbits

For these, $L = 1$ in (3.6) and we again put $\theta=\pi/2$, $\dot{\theta}=0$. The constants of the motion (3.7) still exist for a massive particle, but the r.h.s. of (3.8) becomes $(\dot{\phi})^{-2}$ and as a result (3.10) now reads

$$\left(\frac{du}{d\phi}\right)^2 - \left(A^{2-2a-b}j^2 - A^{2-a-2b}k^2 - A^{2-a-b}\right)\frac{1}{h^2} + Au^2 = 0 \quad . \tag{3.32}$$

Differentiating this w.r.t. ϕ and using a prime to denote the derivative, we get

$$u'u'' + uu' = \frac{Mu'}{h^2}\left[(2-a-b)A^{1-a-b} + (-2+2a+b)A^{1-2a-b}j^2\right.$$

$$\left. + (2-a-2b)A^{1-a-2b}k^2\right] + 3Mu^2u' \quad . \tag{3.33}$$

This is of course trivially satisfied for circular orbits. But for non-circular orbits we can use it to work out the perihelion problem following the same method as in 4D (see, e.g., Adler, Bazin and Schiffer 1975). Thus we cancel u', take the weak-field case $Mu = M/r << 1$ and neglect terms of order $(Mu)^2$ and higher, to obtain

$$u'' + (1+\varepsilon\gamma)u = \eta + \frac{\varepsilon u^2}{\eta} \quad . \tag{3.34}$$

Here we have introduced 3 constants, namely ε, γ and η. These are related via 2 intermediate constants to the parameters a, b of the metric (3.4) and the constants of the motion *j, h, k* of (3.4):

$$\varepsilon \equiv \frac{3M^2d}{h^2} << 1 \qquad \gamma \equiv -\frac{e}{3d} \qquad \eta \equiv \frac{Md}{h^2}$$

$$d \equiv (1+k^2) + (a-1)(-1+2j^2-k^2) + b(-1+j^2-2k^2)$$

$$e \equiv 2(2-a-b)(-1+a+b) + 2j^2(-2+2a+b)(-1+2a+b) + 2k^2(2-a-2b)(-1+a+2b) \,. \tag{3.35}$$

The solution of (3.34) to first order in ε is

$$u = \frac{1}{r} = \eta + \varepsilon\eta(1-\gamma) + \varepsilon\frac{C^2}{2\eta} - \varepsilon\frac{C^2}{6\eta}\cos 2\phi + \left(1-\frac{\gamma}{2}\right)C\cos\left\{\left[1-\varepsilon\left(1-\frac{\gamma}{2}\right)\right]\phi\right\} \tag{3.36}$$

where C is an integration constant. The perihelion shift arises from the lack of periodicity exhibited by the last equation. Its value between successive perihelia amounts to

$$\delta\phi = 2\pi\varepsilon\left(1-\frac{\gamma}{2}\right) = \frac{6\pi M^2}{h^2}\left(d+\frac{e}{6}\right) \,. \tag{3.37}$$

This expression can be simplified if we put $k = 0$ and use preceding relations (Kalligas, Wesson and Everitt 1995). Thus the perihelion shift for a massive particle in a nearly circular orbit in 3-space that has no velocity in the extra dimension is

$$\delta\phi = \frac{6\pi M}{r}\left(a + \frac{2}{3}b\right) \ . \tag{3.38}$$

This of course gives back the standard value in the Schwarzschild case with $a = 1$, $b = 0$. But even a small departure of the extra dimension from flatness $(b \neq 0)$ affects the first or dominant part of the metric $(a \neq 1)$, so other values are possible.

Another type of motion for massive particles that is of interest is radial free fall. In Newton's theory, a particle's speed constantly increases and becomes infinite for $r = 0$. In Einstein's theory, however, the particle's coordinate speed actually starts to decrease at $r = 6M$ and becomes zero on the Schwarzschild surface $r = 2M$. In Kaluza-Klein theory, we expect the motion of a particle falling in the soliton field (3.4) to be different again. To investigate this, we put $\dot{\theta} = 0 = \dot{\phi}$ in (3.6), which now reads

$$A^a \dot{t}^2 - A^{-(a+b)}\dot{r}^2 - A^b \dot{l}^2 = 1 \ . \tag{3.39}$$

This with (3.7) can be rewritten as

$$\dot{r}^2 + A^a k^2 - A^b j^2 + A^{a+b} = 0 \ . \tag{3.40}$$

For a particle at rest at $r = r_0$ this gives

$$j^2 = \left[A(r_0)\right]^{a-b} k^2 + \left[A(r_0)\right]^a \ . \tag{3.41}$$

Putting this back into (3.40) makes the latter read

$$\dot{r}^2 = \left[\left(1 - \frac{2M}{r}\right)^b \left(1 - \frac{2M}{r_0}\right)^{a-b} - \left(1 - \frac{2M}{r}\right)^a\right] k^2 + \left(1 - \frac{2M}{r}\right)^b \left(1 - \frac{2M}{r_0}\right)^a - \left(1 - \frac{2M}{r}\right)^{a+b} \ . \tag{3.42}$$

This can be thought of as an energy equation for our problem. Indeed, it is easily seen that in the limit where $a = 1$, $b = 0$, $k = 0$ it reduces to $\dot{r}^2/2 = M(1/r - 1/r_0)$, which is the corresponding expression for the Schwarzschild case in Einstein's theory and has the same form as the energy equation in Newton's theory. In Kaluza-Klein theory, the coordinate velocity in the r direction is

$$\frac{dr}{dt} = \frac{dr}{dS}\frac{dS}{dt} . \tag{3.43}$$

The second factor here can be written using (3.7) and (3.41) as

$$\frac{dS}{dt} = \left(1-\frac{2M}{r}\right)^a \Bigg/ \sqrt{\left(1-\frac{2M}{r_0}\right)^{a-b} k^2 + \left(1-\frac{2M}{r_0}\right)^a} . \tag{3.44}$$

The first factor can be written using (3.42) as

$$dr/dS = -\Big\{\Big[(1-2M/r)^b(1-2M/r_0)^{a-b} - (1-2M/r)^a\Big]k^2$$

$$+(1-2M/r)^b(1-2M/r_0)^a - (1-2M/r)^{a+b}\Big\}^{1/2} . \tag{3.45}$$

The combination means (3.43) becomes

$$\frac{dr}{dt} = -\left(1-\frac{2M}{r}\right)^a \left\{\left[\left(1-\frac{2M}{r}\right)^b\left(1-\frac{2M}{r_0}\right)^{a-b} - \left(1-\frac{2M}{r}\right)^a\right]k^2\right.$$

$$\left.+\left(1-\frac{2M}{r}\right)^b\left(1-\frac{2M}{r_0}\right)^a - \left(1-\frac{2M}{r}\right)^{a+b}\right\}^{1/2} \Bigg/ \left[\left(1-\frac{2M}{r_0}\right)^{a-b} k^2 + \left(1-\frac{2M}{r_0}\right)^a\right]^{1/2} . \tag{3.46}$$

For a particle starting at an infinite radial distance from the central body, this implies

$$\frac{dr}{dt} = -\left(1-\frac{2M}{r}\right)^a \left\{\left[\left(1-\frac{2M}{r}\right)^b - \left(1-\frac{2M}{r}\right)^a\right]k^2 + \left(1-\frac{2M}{r}\right)^b\left[1-\left(1-\frac{2M}{r}\right)^a\right]\right\}^{1/2} \Bigg/ \sqrt{1+k^2}$$

. (3.47)

One feature of this result is that the radial speed of a test particle will depend on its velocity component along the fifth dimension (through k). In situations where the latter cannot be

neglected, a distant observer may interpret this dependence as an apparent violation of the equivalence principle. This is because different test particles may have different values of k and therefore different values of dr/dt. [It is easy to see, after multiplying equation (3.47) by $(g_{rr})^{1/2}/(g_{tt})^{1/2} = A^{-(a+b/2)}$, that a similar statement holds true for the locally measured radial speed.] For cases of negligible k a straightforward yet rather lengthy calculation shows that the radius where $|dr/dt|$ starts decreasing is given by

$$r_* = 2M \Big/ \left[1 - \left(\frac{2a+b}{3a+b}\right)^{1/a}\right] . \tag{3.48}$$

This for $a = 1$, $b = 0$ gives back the $r_* = 6M$ of the 4D Schwarzschild solution. But clearly the dynamics of the central regions of 5D solitons in general differs from that near 4D black holes, even for $k = 0$. For $k \neq 0$ the situation is even more complicated, and we will return to the issue of apparent violations of the equivalence principle in Section 3.7 below.

3.5 The Redshift Effect

The shift in frequency of electromagnetic waves in the gravitational field of the soliton (3.4) is easy to treat because the latter is static. The frequency of received and emitted radiation is given as usual by

$$\frac{\nu_r}{\nu_e} = \left[\frac{g_{00}(x_e)}{g_{00}(x_r)}\right]^{1/2} . \tag{3.49}$$

For the radial case in the weak-field limit this gives

$$\frac{\nu_r}{\nu_e} = \left[\left(1 - \frac{2M}{r_e}\right) \Big/ \left(1 - \frac{2M}{r_r}\right)\right]^{a/2} \cong 1 + aM\left(\frac{1}{r_r} - \frac{1}{r_e}\right) , \tag{3.50}$$

which can be written as

$$\frac{\nu_r - \nu_e}{\nu_e} = aM\left(\frac{1}{r_r} - \frac{1}{r_e}\right) . \tag{3.51}$$

The corresponding expression for the 4D Schwarzschild metric is

$$\frac{\nu_r - \nu_e}{\nu_e} = M\left(\frac{1}{r_r} - \frac{1}{r_e}\right) , \tag{3.52}$$

so it is natural to identify the gravitational mass for the 5D soliton metric as

$$M_g = aM . \tag{3.53}$$

This means that the gravitational mass, which figures in astronomical relations such as Kepler's laws, is the same as in general relativity not only for the Schwarzschild case ($a = 1$, $b = 0$) but also for the Chatterjee case ($a = 1$, $b = -1$: Chatterjee 1990). However, we will see in Section 3.7 below that in general the gravitational and inertial masses of a soliton are not the same.

3.6 The Geodetic Effect and GP-B

Another possible test of general relativity which has been discussed for many years concerns the geodetic precession of the axis of a spinning object, such as a gyroscope, as it moves through curved spacetime. It is feasible to carry out this test using superconducting technology and a satellite in Earth orbit, as in the Gravity-Probe B mission. (See the volume edited by Jantzen, Keiser and Ruffini 1996 for reviews of various aspects of this.) It is clearly of great importance to find out if there is any predicted difference in this effect for 4D versus 5D.

Consider therefore a spacelike spin vector S^A and a timelike orbit $x^A = x^A(S)$ in the soliton field (3.4). We wish to compute the change in the spatial orientation of S^A after parallel transport once around a circular orbit. The requirement of parallel transport implies that

$$\frac{dS^C}{dS} + \Gamma^C_{AB} S^A \dot{x}^B = 0 , \tag{3.54}$$

where as above an overdot denotes differentiation with respect to S, which should not be confused with the components of S^A. Since the latter is spacelike and $\dot{x}^B$ is timelike, their inner product can be made to vanish:

$$g_{AB}S^A\dot{x}^B = 0 \quad . \tag{3.55}$$

Our objective is to solve (3.54) and (3.55) for the S^A as functions of S. In order to simplify the calculation we will assume that $S^4 = 0$. Since $x^A = x^A(S)$ is a circular orbit, we can write for $\dot{x}^A$

$$\dot{x}^A(S) = \left(i,\, 0,\, 0,\, \frac{d\phi}{dt}\, i,\, 0 \right). \tag{3.56}$$

Expressions for i and $\Omega \equiv d\phi / dt$ have been worked out by Kalligas, Wesson and Everitt (1995). They are:

$$i = \left(1 - \frac{2M}{r}\right)^{-a/2} \left[1 - \frac{aM}{r - (1+a+b)M}\right]^{-1/2} \tag{3.57}$$

$$\Omega = \left[\frac{(1-a-b)}{a}\left(1 - \frac{2M}{r}\right)^{1-2a-b} r^2 + \frac{1}{aM}\left(1 - \frac{2M}{r}\right)^{2-2a-b} r^3\right]^{-1/2} . \tag{3.58}$$

Now (3.56) in (3.55) causes the latter to become

$$g_{00}S^0\dot{x}^0 + g_{33}S^3\dot{x}^3 = 0\,, \tag{3.59}$$

which in view of (3.4) reads

$$S^0 = (1 - 2M/r)^{1-2a-b}\,\Omega r^2 S^3 \quad . \tag{3.60}$$

This relation between S^0 and S^3 makes the system of equations (3.54) redundant, since the $A = 0$ and $A = 3$ members are the same. Furthermore, our assumption that $S^4 = 0$ together with (3.56) and the expressions (3.5) for the Christoffel symbols, makes the $A = 4$ member of (3.54) trivial. Hence we are left with 3 of 5 of the equations (3.54):

$$\frac{dS^1}{dS}+\Gamma^1_{00}S^0\dot{x}^0+\Gamma^1_{33}S^3\dot{x}^3=0$$

$$\frac{dS^2}{dS}=0$$

$$\frac{dS^3}{dS}+\Gamma^3_{13}S^1\dot{x}^3=0 \quad . \tag{3.61}$$

These with (3.56) and (3.5) imply

$$\frac{dS^1}{dS}-\left[\frac{r-(1+a+b)M}{(1-2M/r)^a}\right]\frac{\Omega}{i}S^3=0$$

$$S^2 = \text{constant}$$

$$\frac{dS^3}{dS}+\left[\frac{r-(1+a+b)M}{r(r-2M)}\right]\Omega i S^1=0 \quad . \tag{3.62}$$

The solution of this system of equations is given by the following:

$$S^1=\frac{H}{r}\left\{[r(r-2M)]\left[\frac{r-(1+2a+b)M}{r-(1+a+b)M}\right]\right\}^{1/2}$$

$$\times\cos\left\{\phi_0-r^{(a-1)/2}[r-(1+a+b)M](r-2M)^{-(a+1)/2}S\Big/\left[\frac{(1-a-b)}{a}A^{1-2a-b}r^2+A^{2-2a-b}\frac{r^3}{Ma}\right]^{1/2}\right\}$$

$$S^2 = \text{constant}$$

$$S^3=\frac{H}{r}\sin\left\{\phi_0-r^{(a-1)/2}[r-(1+a+b)M](r-2M)^{-(a+1)/2}S\Big/\left[\frac{(1-a-b)}{a}A^{1-2a-b}r^2+A^{2-2a-b}\frac{r^3}{Ma}\right]^{1/2}\right\} \quad . \tag{3.63}$$

Here H and ϕ_0 are constants. We can see from these equations and (3.58) that the spatial part of S^A rotates relative to the radial direction with proper angular speed Ω_S given by

$$\Omega_S=\frac{[1-(1+a+b)M/r]}{(1-2M/r)^{(a+1)/2}}\,\Omega \quad . \tag{3.64}$$

This with $a \to 1$, $b \to 0$ gives $\Omega_s \to \Omega$, the usual result of 4D general relativity. We also see that if S^A were originally radially oriented, then upon completion of one revolution along $x^A(S)$ its direction would be shifted from the radial by

$$\delta\phi = 2\pi\left[1 - \frac{\sqrt{r-(1+a+b)M}\ \sqrt{r-(1+2a+b)M}}{\sqrt{r(r-2M)}}\right]. \tag{3.65}$$

In the weak-field limit this expression reduces to

$$\delta\phi = \frac{(3a+2b)\pi M}{r}. \tag{3.66}$$

This is the geodetic precession of the axis of a spinning object in the soliton field (3.4), and while agreeing with the usual 4D result for $a = 1$, $b = 0$ will in general differ from it.

The result (3.66) is simple and in principle offers an experimental way to distinguish between 4D and 5D. For example, the GP-B mission involves a gyroscope in a 650-km altitude Earth orbit and is highly sensitive to differences between general relativity and alternative theories of gravity. As an illustration, let us assume that a, b for the Earth have the values used above for the (borderline detectable) light-deflection case, namely $b = 0.0075$, $a = 0.9962$. Then (3.66) gives $\delta\phi \cong 1.23$ milliarcseconds per revolution, or an annual rate of 6.″633/year. This can be compared to the usual value for 4D general relativity, which corresponds to closing the orbit in a geodetic advance of the spin vector [see (3.66) with $b = 0$, $a = 1$] and is 6.″625/year. Such a difference should be detectable by GP-B.

3.7 The Equivalence Principle and STEP

A further test of general relativity, as opposed to other theories of gravity, concerns the level to which we can expect absolute accordance with the (weak) equivalence principle. The latter in its shortest form says that gravitational mass and inertial mass are the same, so that

objects of differing natures fall at the same rate in a gravitational field (see Section 1.3). It would not be appropriate to review here the large literatures that exist on the theoretical and experimental sides of this issue. But we have already in Section 3.4 come upon dynamical effects of 5D theory which appear to constitute apparent violations of the equivalence principle, albeit at levels which are probably very low. Obviously, if there is any possibility that this principle might not be obeyed, it should be continuously tested to even more rigorous levels. Proposals to do this are frequent in gravitational physics. Of these, most are impractical, though some are redeemed by imagination (e.g., Wit 1991). A practical test which promises unprecedented accuracy involves measuring the accelerations of different test masses aboard a satellite in Earth orbit, as in the Satellite Test of the Equivalence Principle mission. (See the volume edited by Reinhard 1996 for reviews of various aspects of this.) In view of the extensive literature that already exists on the topic, we will here content ourselves with a short historical account of possible violations of the 4D equivalence principle in 5D gravity, and afterwards make some pertinent comments about STEP and related tests.

Gross and Perry (1983) evaluated the inertial mass of a soliton by using the Landau pseudo energy-momentum tensor, and the gravitational mass by using the metric. They found $M_i = (1+\beta)M_g$, so the masses are different for any finite value of the parameter β, which by (3.3) fixes the potential associated with the fifth dimension. Only when the latter has no effect and $\beta = 0$ do we recover the Schwarzschild solution with $M_i = M_g$. However, while it may appear that $\beta \neq 0$ involves a violation of the equivalence principle, Gross and Perry argued that the source of the discrepancy is actually just the breakdown of Birkhoff's theorem.

Deser and Soldate (1989) agreed with this analysis, though they noted that the precise relation between m_i and m_g needs correcting by a factor two from the formula in Gross and

Perry (1983). We have reworked the calculation involving the Landau tensor and confirmed that $M_i = (1+\beta/2)M_g$, as well as studying the motion of a test particle near a soliton (Billyard, Wesson and Kalligas 1995). The important thing is that in 5D there are two different masses, a property which inevitably makes the dynamics of Kaluza-Klein theory different from that of Einstein theory.

Cho and Park (1991) evaluated the inertial and gravitational masses of a soliton not in 5D but in 7D. They found them to be unequal, and at first stated that this indicated "an obvious violation of the equivalence principle". However, in a subsequent discussion they argued that the extra or fifth force associated with the extended geometry is responsible for this apparent violation, and that it should be described not as a breakdown of the equivalence principle but as a departure from the usual (4D) law of geodesic motion.

Wesson (1996) reviewed these arguments, and by using equations of motion like those derived in Section 3.4 above, came to the conclusion that the problem is largely one of semantics: the relations that describe geodesic motion in 5D are not in general the same as those which describe geodesic motion in 4D, and the extra terms from 5D look like violations of the equivalence principle in 4D.

We will make a detailed examination of geodesic motion in 5D later, where we will find that there are some 5D metrics where the 4D part of the 5D motion is indistinguishable from pure 4D motion, and some metrics which imply significant differences. Here, it is worthwhile to note that missions like STEP can in general expect two types of practical difference between 4D and 5D physics. First, the orbit of the spacecraft as a whole will not be the same in Einstein theory and Kaluza-Klein theory, and since the mission involves altitude-related telemetry the difference may be detectable. Second, the accelerations of the test masses aboard the spacecraft

will not be the same if their inertial and gravitational masses differ as in Kaluza-Klein theory, and since these can be measured to high accuracy a difference may be detectable.

3.8 Conclusion

The 4D Schwarzschild metric (3.1) is embedded in the 5D soliton metrics (3.3) and (3.4) via a choice of parameters that effectively nullifies the influence of the fifth dimension. The question of whether we need the latter is in principle answerable by observation and experiment. Thus (3.18) gives the light-deflection angle, (3.25) gives the radar-ranging time delay, (3.31) gives the circular photon orbit radius, (3.38) gives the perihelion advance angle, (3.48) gives the motion inversion radius, (3.51) gives the shift in radiation frequency, and (3.66) gives the geodetic advance angle. Data from the solar system (Will 1992) indicate that the dimensionless constant which measures the influence of the fifth potential is at most of order 10^{-3} locally.

It is of course a paradigm of current research methodology that we should embed what we know in what we do not know, smoothly. This is really just a modern version of Occam's razor, and as such is laudable. But because of the finite uncertainties that attend all observations this philosophy necessarily leads to a situation where it is difficult to validate new theory using old data.

The way forward is to look at systems other than the solar system and to carry out new tests in space. Specifically, we need to test the light-deflection formula by better observations of lensed QSOs and other objects (Kochanek 1993; Gould 1994). And we need to test the perihelion-advance formula by better observations of eclipsing binaries such as DI Herculis (Guinan and Maloney 1985; Maloney, Guinan and Boyd 1989), binary pulsars (Haugan 1985; Taylor and Weisberg 1989) and possible pulsars with planetary comparisons (Wolszczan and Frail 1992; Thorsett and Phillips 1992). Also, we need to test the geodetic-precession formula

by carrying out missions like GP-B, and we need to test the equality of gravitational and inertial mass by carrying out missions like STEP.

References (Chapter 3)

Adler, R., Bazin, M. Schiffer, M. 1975, Introduction to General Relativity (2nd ed., McGraw-Hill, New York).

Billyard, A., Wesson, P.S., Kalligas, D. 1995, Int. J. Mod. Phys. D4, 639.

Chatterjee, S. 1990, Astron. Astrophys. 230, 1.

Cho, Y.M., Park, D.H. 1991, Gen. Rel. Grav. 23, 741.

Davidson, D., Owen, D. 1985, Phys. Lett. B 155, 247.

Deser, S., Soldate, M. 1989, Nucl. Phys. B 311, 739.

Gould, A. 1994, Astrophys. J. 435, 573.

Gross, D.J., Perry, M.J. 1983, Nucl. Phys. B 226, 29.

Guinan, E.F., Maloney, F.P. 1985, Astron. J. 90, 1519.

Haugan, M.P. 1985, Astrophys. J. 296, 1.

Jantzen, R.T., Keiser, G.M., Ruffini, R. (eds.) 1996, Proc. 7th Marcel Grossman Meeting (World Scientific, Singapore).

Kalligas, D., Wesson, P.S., Everitt, C.W.F. 1995, Astrophys. J. 439, 548.

Kochanek, C.S. 1993, Astrophys. J. 419, 12.

Lim, P., Overduin, J.M., Wesson, P.S. 1995, J. Math. Phys. 36, 6907.

Maloney, F.P., Guinan, E.F., Boyd, P.T. 1989, Astron. J. 98, 1800.

Reinhard, R. (ed.) 1996, STEP: Testing the Equivalence Principle in Space (European Space Agency, WPP-115).

Taylor, J.H., Weisberg, J.M. 1989, Astrophys. J. 345, 434.

Thorsett, S.E., Phillips, J.A. 1992, Astrophys. J. 387, L69.

Wesson, P.S. 1996, in STEP: Testing the Equivalence Principle in Space (European Space Agency, WPP-115), 566.

Will, C.M. 1992, Int. J. Mod. Phys. D 1, 13.

Wit, P.S. 1991, in Gravitation: A Banff Summer Institute (eds. Mann, R., Wesson, P.S., World Scientific, Singapore), 638.

Wolszczan, A., Frail, D.A. 1992, Nature 355, 145.

4. COSMOLOGY AND ASTROPHYSICS IN 5D

"Was there really a big bang?" (Sir Martin Rees, Toronto, 1997)

4.1 Introduction

In Chapter 2, we learned that cosmological matter with uniform density and pressure can be regarded as induced in 4D from 5D, and gave a general prescription for neutral matter wherein 4D physical quantities are derived from 5D geometrical quantities. There is now a considerable literature on induced-matter theory. In this chapter, however, we will concentrate on solutions of cosmological and astrophysical importance. We will take a closer look at the 'standard' cosmological model, and then examine a model for spherically-symmetric clouds of matter which has general application to astrophysical systems (e.g. clusters of galaxies). These models basically agree with cosmological and astrophysical data because they smoothly embed 4D solutions in 5D solutions. The induced-matter theory goes further, though, and in the last half of this chapter we will look at new types of solutions of cosmological relevance. These include a solution which describes waves in a de Sitter vacuum and has implications for inflation, and a time-dependent version of the static soliton examined before which may help explain how these objects formed. We will also look briefly at systems with axial and cylindrical symmetry, which while of mainly mathematical interest have implications for the (none) inevitability of big-bang singularities. Lastly, we will present a shell-like solution, which while derived for use in particle physics may also be used as a model for the cellular structure which seems to be a common feature of cosmological surveys.

4.2 The Standard Cosmological Model

This was presented in Section 2.2 as the induced-matter version of a 5D solution to $R_{AB} = 0$ derived originally by Ponce de Leon (1988). In the present section we will present it in a new way, using results from coordinate transformations and the geodesic equation (Mashhoon, Liu and Wesson 1994; Wesson 1994a; Wesson and Liu 1995). After some algebraic work, we will make contact with observations via the subjects of dark matter, the 3K microwave background and a possible time-variation of particle masses.

Any solution of 5D $R_{AB} = 0$ can be written as a solution of 4D $G_{\alpha\beta} = 8\pi T_{\alpha\beta}$ with an effective or induced energy-momentum tensor, given by (2.53) as:

$$8\pi T_{\alpha\beta} = \frac{\Phi_{\alpha;\beta}}{\Phi} - \frac{\varepsilon}{2\Phi^2}\left\{\frac{\overset{*}{\Phi}\overset{*}{g}_{\alpha\beta}}{\Phi} - \overset{**}{g}_{\alpha\beta} + g^{\lambda\mu}\overset{*}{g}_{\alpha\lambda}\overset{*}{g}_{\beta\mu} - \frac{g^{\mu\nu}\overset{*}{g}_{\mu\nu}\overset{*}{g}_{\alpha\beta}}{2} + \frac{g_{\alpha\beta}}{4}\left[\overset{*}{g}^{\mu\nu}\overset{*}{g}_{\mu\nu} + \left(g^{\mu\nu}\overset{*}{g}_{\mu\nu}\right)^2\right]\right\} \qquad (4.1)$$

Here as before, $\Phi_\alpha \equiv \partial\Phi / \partial x^\alpha$, a semicolon denotes the 4D covariant derivative, and an asterisk means $\partial/\partial x^4$. The form of (4.1) depends on a gauge or coordinate choice wherein the 5D interval is written $dS^2 = g_{AB}dx^A dx^B$, with a 5D metric tensor $g_{AB} = \left(g_{\alpha\beta},\ g_{4\alpha} = 0,\ g_{44} = \varepsilon\Phi^2,\ \varepsilon = \pm 1\right)$. However, while this gauge suppresses overtly electromagnetic effects, the results it leads to are general in the sense that as long as we allow g_{AB} to be x^4-dependent we have only used 4 of the 5 available coordinate degrees of freedom (see Section 2.5). This leaves 1 over, which we will use below in a way indicated by the history of physical measurements (see Section 1.2). We will use (4.1) above to derive all relevant properties of matter for the 5D solutions presentcd below.

For cosmology, let us start buddhistically with featureless, flat 5D space:

$$ds^2 = dt^2 - dr^2 - r^2 d\Omega^2 - dl^2 \ . \qquad (4.2)$$

Here the coordinates of spacetime are the time (t) and spherical polars $\left(r, \theta, \phi : d\Omega^2 \equiv d\theta^2 + \sin^2\theta\, d\phi^2\right)$, and the extra coordinate is $x^4 = l$. Let us now change coordinates to

$$T = t \qquad R = \frac{r}{l}\left(1 + \frac{r^2}{l^2}\right)^{-1/2} \qquad L = l\left(1 + \frac{r^2}{l^2}\right)^{1/2} \ . \qquad (4.3)$$

Then (4.2) becomes

$$dS^2 = dT^2 - L^2\left[\frac{dR^2}{(1-R^2)} + R^2\, d\Omega^2\right] - dL^2 \ . \qquad (4.4)$$

If we now put this into (4.1) and take the matter to be a perfect fluid $\left(T_0^0 = \rho,\ T_1^1 = -p\right)$, we find that its density and pressure are given by

$$8\pi\rho = \frac{3}{L^2} \qquad 8\pi p = -\frac{1}{L^2} \qquad \rho + 3p = 0 \ : \qquad (4.5)$$

The equation of state here is typical of situations in 4D where the gravitational or Tolman-Whittaker mass is zero (see Section 2.4). Since matter with this equation of state does not disturb other objects gravitationally, it has been suggested as relevant to cosmic strings (Gott and Rees 1987; Kolb 1989), zero-point fields (Wesson 1991, 1992), and extreme sources for the Reissner-Nordstrom metric (Ponce de Leon 1993). However, it is always possible to split the density and pressure up into parts that are due to matter and parts that are due to vacuum, where the latter are related to the cosmological constant via $\rho_v = \Lambda / 8\pi = -p_v$. Thus if we write $\rho = \rho_m + \rho_v$ and $p = p_m + p_v$ in (4.5), introduce Λ and set $p_m = 0$, we obtain

$$8\pi\rho_m = \frac{2}{L^2} \qquad p_m = 0 \qquad \Lambda = \frac{1}{L^2} \ . \qquad (4.6)$$

This is an acceptable prescription for an Einstein universe where the matter is dust supported by a positive cosmological constant.

The same results can naturally be arrived at not by using (4.1) but by using the Einstein equations, since these are a subset of the Kaluza-Klein equations which are satisfied by (4.2) and (4.4). Thus the Robertson-Walker metric in the usual coordinates

$$ds^2 = dt^2 - S^2(t)\left[\frac{dr^2}{(1-kr^2)} + r^2\, d\Omega^2\right] \tag{4.7}$$

matches the metric (4.4) on the hypersurface L = constant with curvature index $k = +1$ and scale factor $S = L$. And the Friedmann equations

$$\begin{aligned} 8\pi\rho_m &= \frac{3k}{S^2} + \frac{3\dot{S}^2}{S^2} - \Lambda \\ 8\pi p_m &= -\frac{k}{S^2} - \frac{\dot{S}^2}{S^2} - \frac{2\ddot{S}}{S} + \Lambda \end{aligned} \tag{4.8}$$

give back the prescription (4.6). The point is that the 4D Einstein universe with reasonable properties of matter can be viewed as a 5D model on a hypersurface whose size is related to the cosmological constant.

Of course, one might argue that this is not physically important because the Einstein universe is static. However, time dependence is easily introduced. Consider, for example, new coordinates t, l (not to be confused with those above) related to T, L by

$$T = l\sinh t \qquad R = r \qquad L = l\cosh t. \tag{4.9}$$

Then (4.4) becomes

$$dS^2 = l^2\, dt^2 - l^2\cosh^2 t\left[\frac{dr^2}{(1-r^2)} + r^2\, d\Omega^2\right] - dl^2\ . \tag{4.10}$$

This into (4.1) gives

$$8\pi\rho=\frac{3}{l^2} \qquad 8\pi p=-\frac{3}{l^2} \qquad \rho+p=0\ . \tag{4.11}$$

The equation of state here is the same as the one for the vacuum in general relativity (see Section 1.3). On the hypersurface $l=$ constant, (4.10) matches (4.7) and describes a vacuum-dominated universe with spherical space sections and no big bang.

The above examples show that from 5D Minkowski space we can obtain a static metric and an expanding metric which correspond to 4D cosmological models on hypersurfaces x^4 = constants. This is perhaps not too remarkable from the mathematical side, because it is possible to embed 4D Robertson-Walker spaces in flat 5D space (Robertson and Noonan 1968; Rindler 1977). However, this involves a constraint between the five coordinates which is generally different from the one (x^4 = constant) considered above. And from the physical side, x^4-dependent 5D metrics lead to different equations of state as in (4.5) and (4.11), which implies that the search for cosmological models that describe the real universe is also a search for appropriate coordinate systems.

An appropriate system for cosmology is one that puts the metric into the form

$$dS^2=l^2dt^2-t^{2/\alpha}l^{2/(1-\alpha)}\left[dr^2+r^2d\Omega^2\right]-\alpha^2(1-\alpha)^{-2}t^2dl^2\ . \tag{4.12}$$

Here α is a constant related to the properties of matter, and we gave the explicit coordinate transformation that connects (4.2) and (4.12) in (2.11). The coordinates here should again not be confused with any used above, and are in fact just the ones used to ensure that the 5D models reduce to the 4D (space-flat) FRW ones on hypersurfaces x^4 = constants (Ponce de Leon 1988). The density and pressure of the cosmological fluid can be obtained by putting (4.12) into (4.1), which gives

$$8\pi\rho=\frac{3}{\alpha^2l^2t^2} \qquad 8\pi p=\frac{(2\alpha-3)}{\alpha^2l^2t^2} \qquad p=\left(\frac{2\alpha-3}{3}\right)\rho\ . \tag{4.13}$$

These are as derived before by a different method (Section 2.3); and as noted, there are two cases which are of special physical interest.

For $\alpha = 3/2$ we have dust with distances that increase in proportion to $t^{2/3}$ as in the matter-dominated (Einstein-de Sitter) model. For $\alpha = 2$ we have $p = \rho/3$ with distances that increase in proportion to $t^{1/2}$ as in the radiation-dominated model. Thus (4.12) and (4.13) give back the geometry and properties of matter of the standard $k = 0$ FRW models for the present and early universe.

The standard 5D model contains the standard 4D model, and observations that support the latter also support the former. However, there is a well-known problem with the 4D Einstein-de Sitter model, which is that it implies a density of dark matter far in excess of what is present in luminous galaxies. The nature of this dark matter is unknown. We mentioned this issue in Section 1.6, where we noted that weakly-interacting massive particles of the kind predicted by supersymmetry are a viable candidate from the perspective of particle physics. However, solitons are a viable candidate from the perspective of classical physics. Indeed, as the 5D analogs of 4D black holes we would expect them to be ubiquitous.

We made an in-depth study of the soliton metric in Section 2.4, but it is convenient to restate it here:

$$dS^2 = \left(\frac{ar-1}{ar+1}\right)^{2\varepsilon\kappa} dt^2 - \left(\frac{a^2r^2-1}{a^2r^2}\right)^2 \left(\frac{ar+1}{ar-1}\right)^{2\varepsilon(\kappa-1)} \left[dr^2 + r^2 d\Omega^2\right] - \left(\frac{ar+1}{ar-1}\right)^{2\varepsilon} dl^2. \tag{4.14}$$

Here ε, κ are dimensionless constants which obey the consistency relation $\varepsilon^2\left(\kappa^2 - \kappa + 1\right) = 1$. The choice $\varepsilon \to 0$, $\kappa \to \infty$, $\varepsilon\kappa \to 1$ gives back the 4D Schwarzschild solution plus an extra flat dimension. In this limit we identify the dimensional constant in (4.14) as $a = 2/M_*$ where M_* is

the mass of an object such as a star. The metric (4.14) is an exact solution of the 5D Kaluza-Klein field equations $R_{AB} = 0$, so its induced matter can be calculated from (4.1):

$$8\pi T_0^0 = \frac{4\varepsilon^2 \kappa a^6 r^4}{(ar-1)^4(ar+1)^4}\left(\frac{ar-1}{ar+1}\right)^{2\varepsilon(\kappa-1)}$$

$$8\pi T_1^1 = \frac{4\varepsilon a^5 r^3}{(ar-1)^3(ar+1)^3}\left(\frac{ar-1}{ar+1}\right)^{2\varepsilon(\kappa-1)} - \frac{4\varepsilon a^6 r^4(2\varepsilon+2ar-\varepsilon\kappa)}{(ar-1)^4(ar+1)^4}\left(\frac{ar-1}{ar+1}\right)^{2\varepsilon(\kappa-1)}$$

$$8\pi T_2^2 = -\frac{2\varepsilon a^5 r^3}{(ar-1)^3(ar+1)^3}\left(\frac{ar-1}{ar+1}\right)^{2\varepsilon(\kappa-1)} - \frac{4\varepsilon a^6 r^4(\varepsilon\kappa-\varepsilon-ar)}{(ar-1)^4(ar+1)^4}\left(\frac{ar-1}{ar+1}\right)^{2\varepsilon(\kappa-1)}$$

$$T_3^3 = T_2^2 \quad . \tag{4.15}$$

These components agree with those arrived at by a somewhat different method in Section 2.4. They satisfy $T_0^0 + T_1^1 + T_2^2 + T_3^3 = 0$, so the equation of state is radiation-like. We can write this as $\bar{p} = \rho/3$, where $\bar{p} \equiv -\left(T_1^1 + T_2^2 + T_3^3\right)/3$ is the averaged pressure, but in general $T_1^1 \neq T_2^2$ in (4.15) so the fluid is anisotropic. The gravitational mass of this fluid out to radius r is given by $M_g = (2\varepsilon\kappa/a)\left[(ar-1)/(ar+1)\right]^{\varepsilon}$ by (2.30). We refer back to Chapters 2 and 3 for a discussion of other properties of (4.14), and conclude this recap by noting that solitons for general choices of the parameters ε, κ are distended spheres of radiation-like matter.

These 5D fireballs can, however, have low temperatures. Solitons are classical or macroscopic objects, whereas temperature is conventionally defined by particle or microscopic effects. The same applies to the spectrum of the radiation they involve. Therefore, it is quite possible that the universe is populated by cool solitons (Wesson 1994b). We proceed to discuss 3 ways that can be used to detect and constrain them.

Background radiation can be significantly affected by solitons, because by (4.15) they have extended matter. The length scale of the distribution is set by the dimensional constant a in (4.14), which while it is related to the Schwarzschild mass in the 4D limit should be regarded as a geometrical parameter of more generality in the 5D case. For $r >> 1/a$, $\rho \cong \varepsilon^2 \kappa / 2\pi a^2 r^4$ and $M_g \cong 2\varepsilon\kappa / a$. Combining these expressions and restoring conventional dimensions gives the density at large distances as

$$\rho \cong \frac{GM_g^2}{8\pi c^2 \kappa r^4} . \tag{4.16}$$

The parameter κ here is free to choose physically, but a particularly convenient choice for the purpose of illustration is κ = 1. Let us take this and assume that solitons of mass M. are uniformly distributed with mean density $\bar{\rho}$, so that the average separation is of order $d = (M_\bullet / \bar{\rho})^{1/3}$. Then the energy density of the nearest soliton is given by (4.16) with $r = d$, and is

$$E = \rho c^2 \cong \frac{GM_\bullet^{2/3} \bar{\rho}^{4/3}}{8\pi} . \tag{4.17}$$

This is actually sufficient to work out the expected energy density of a collection of solitons, because in the uniform case the volume of space varies as r^3 while the energy density of a given soliton varies as r^{-4}, so the nearest one dominates. The equation of state of the matter of a soliton is radiation-like as we saw above, so we shall assume that the matter is indeed photons. Unfortunately, we do not have spectral information on these (which would come from a detailed model), and so are obliged to proceed bolometrically. However, we can obtain an interesting result by considering a cosmological medium of solitons and comparing with the cosmic microwave background. Thus, let us consider a medium consisting of solitons with galactic

mass $M_\bullet = 5 \times 10^{11}\ M_\odot$ and mean density $\bar{\rho} = 1 \times 10^{-29}$ g cm^{-3}. Then (4.17) shows that the nearest soliton would produce a contribution which compared to that of the uniform background would be of order 10^{-5}. Since the Cosmic Background Explorer (COBE) and other instruments have restricted anomalous contributions to this order, we infer that if the dark matter consists of solitons then they are probably less massive than galaxies.

Galactic morphology can be used to bolster this inference. It is well known that objects consisting of conventional dark matter must be less than of order $10^8\ M_\odot$ if the ones near to regular galaxies are not to affect the shapes of the latter through tidal interactions. A similar argument can be applied to solitons. However, it is not so straightforward. In our preceding discussion of background radiation, we were concerned with one soliton and chose a convenient value of κ for it ($\kappa = 1$, which by the consistency relation implies $\varepsilon = 1$ also). For interacting populations of galaxies and solitons, however, the latter would presumably have a range of values of ε and κ. And some values of these parameters can make the effective masses of solitons quite small. For example, $\kappa = 0.1$ implies that $\varepsilon = 1.05$, so for $r >> 1/a$ we have $M_g = 0.105(2/a) = 0.105\ M_\bullet$ and an effective mass only one-tenth of what it would be conventionally. (The extreme case $\varepsilon\kappa \to 0$ corresponds to an object with zero gravitational mass which is the original Gross and Perry soliton.) We infer that while it is certainly possible, it will take some care to constrain solitons via their effects on the morphologies of regular galaxies.

Gravitational lensing is a more clear-cut effect. Let us assume that a light ray originates in a distant source, suffers a slight deflection δ as it passes near a soliton, and arrives at us. This problem was considered in detail in Chapter 3. There we introduced a quasi-Schwarzschild coordinate $r' = r(1 + 1/ar)^2$ more suited to observations than the isotropic r of (4.14). We also

introduced 3 constants of the motion, which we here relabel k_1, k_2, k_3 and use to define a new constant $d \equiv (2\varepsilon k - \varepsilon - 2)k_1^2 + (2\varepsilon - \varepsilon k + 2)k_3^2$. Then the deflection angle with conventional dimensions restored is given by

$$\delta = \frac{4GM_*}{c^2 r'_{\min}} + \frac{2GM_* dr'_{\min}}{c^2 k_2^2} \quad . \tag{4.18}$$

Here $r'_{\min}$ is the minimum distance of passage of light to the soliton (or impact parameter), and the first term is the usual one found in 4D theory. The second term is special to 5D theory, and is in principle measurable. It is possible to use the lensing of light from QSOs to look for extragalactic solitons and microlensing to look for solitons associated with the Galaxy.

The preceding discussion on dark matter touched at several places on background radiation. This includes the integrated radiation from stars in galaxies at various wavelengths and the contribution from dark-matter particles which are unstable to decay. (It is generally acknowledged that constraints on the so-called extragalactic background light are an effective way to restrict dark-matter particles). However, it also includes the 3K background radiation, whose spectrum has been established by COBE and other instruments to be accurately black body, and is commonly regarded as having been produced by the big bang. Also, nucleosynthesis of the elements is supposed to have occurred in the fireball that followed the big bang. However, there is a problem of logic involved here. This has been realized by a few workers (notably Dicke, Dirac, and Hoyle and Narlikar: see Wesson 1978 for a review). But while it is not generally appreciated, it is deadly.

In 4D, the FRW solutions with metric (4.7) are typically written in comoving coordinates. That is, the solutions are derived under the assumption (which is mathematically valid) that there is a choice of coordinates where $u^{123} = 0$ for the 3 spatial components of the 4-velocity

$u^\alpha \equiv dx^\alpha / ds$. In this frame, galaxies do not move and always retain the same separations from each other. So where (in a physical sense) are the 3K background and nucleosynthesis supposed to fit in? The naive response is that the comoving frame is not 'real', and that in another frame the galaxies are 'really' moving apart in accordance with Hubble's law, and that run backwards in time the universe in this non-comoving frame would at early times produce the extreme conditions necessary to account for the generation of the background radiation and the elements. Such a rationalization has no justification in mathematics or logic. This because, according to 4D general relativity and the principle of (coordinate) covariance on which its equations are based, the comoving frame is just as valid as the non-comoving frame. A little thought shows that this conundrum has a simple solution: 4D algebraic covariance is incomplete with regard to physical processes.

In 5D, this problem can in principle be solved (Wesson and Liu 1995). The fulcrum of the argument involves the physical meaning of the coordinate $x^4 = l$ used in the metric of the standard cosmological model (4.12). This metric has comoving spatial coordinates and on the hypersurface x^4 = constant has an interval given by $dS = ldt$. This should be consistent with 4D particle dynamics, whose corresponding interval or action is defined by $ds = mdt$. Clearly in this case at least, $x^4 = l$ has the physical interpretation of particle rest mass. This agrees with the algebraic possibility of geometrizing mass (Chapter 1), and with the fact that in general x^4-independent metrics describe fluids consisting of photons whereas x^4-dependent metrics describe fluids consisting of massive particles (Chapter 2). Indeed, there is a form for the 5D metric which because of the simplification for particle motion which it implies is called canonical (Mashhoon, Liu and Wesson 1994); and it can be shown that if we wish to give a geometric description of the rest mass m of a particle in 4D Einstein theory, then for canonical metrics in

5D Kaluza-Klein theory the appropriate identification is $l = m$ (Chapter 7). There is a surprisingly extended history of attempts to relate the extra coordinate of 5D relativity to mass, and what we have stated here is only designed to open a prospect to a solution of the covariance problem outlined in the previous paragraph. That this is possible is clear if we recall that we have 1 coordinate degree of freedom left over to use as we wish. Depending on what we choose, we can have a frame where x^4 and masses are constants, or a frame where x^4 and masses are variable. The former is of course the one which has been used in physics historically, but the latter is also acceptable and leads to new insights cosmologically.

To see what is involved here, let us consider the 5D geodesic equation

$$\frac{dU^C}{dS} + \Gamma^C_{AB} U^A U^B = 0 \,. \tag{4.19}$$

Here the 5-velocities are related to the 4-velocities via $U^A \equiv dx^A / dS = \left(dx^A / ds\right)\left(ds / dS\right) = u^\alpha \left(ds / dS\right)$ when $A = \alpha = 0,123$. To get the other component, $U^4 = dx^4 / dS$, we need the following 5D Christoffel symbols for metric (4.12):

$$\Gamma^0_{04} = \frac{1}{l} \qquad \Gamma^0_{44} = \frac{\alpha^2}{(1-\alpha)^2}\frac{t}{l^2}$$

$$\Gamma^4_{00} = \frac{(1-\alpha)^2}{\alpha^2}\frac{l}{t^2} \qquad \Gamma^4_{04} = \frac{1}{t} \quad . \tag{4.20}$$

Then expanding (4.19) shows that its spatial components are satisfied with comoving coordinates of the conventional type $(U^1 = U^2 = U^3 = 0)$, while the zeroth and fourth components yield

$$\frac{dU^0}{dS} + \frac{2}{l}U^0 U^4 + \frac{\alpha^2}{(1-\alpha)^2}\frac{t}{l^2}U^4 U^4 = 0$$

$$\frac{dU^4}{dS} + \frac{(1-\alpha)^2}{\alpha^2}\frac{l}{t^2}U^0 U^0 + \frac{2}{t}U^0 U^4 = 0 \,. \tag{4.21}$$

A solution of these has to be compatible with the condition $g_{AB}U^A U^B = 1$ set by the metric, which is

$$l^2 U^0 U^0 - \frac{\alpha^2}{(1-\alpha)^2} t^2 U^4 U^4 = 1 . \tag{4.22}$$

Equations (4.21), (4.22) do not constrain the signs of the velocities, which turn out to be

$$U^0 = \mp \frac{\alpha}{\sqrt{2\alpha - 1}} \frac{1}{l} \qquad U^4 = \pm \frac{(1-\alpha)^2}{\alpha\sqrt{2\alpha - 1}} \frac{1}{t} . \tag{4.23}$$

The ratio of these gives us a simple equation for $dl/dt = U^4 / U^0$ which in terms of a constant (t_0) yields

$$l = \left(\frac{t_0}{t}\right)^A \qquad A \equiv \left(\frac{1-\alpha}{\alpha}\right)^2 . \tag{4.24}$$

For $\alpha = 2$ (radiation-dominated era) $A = 0.25$, while for $\alpha = 3/2$ (matter-dominated era) $A \cong 0.11$. The relative rate of change of the extra coordinate .

$$\frac{dl/dt}{l} = -\frac{A}{t} \tag{4.25}$$

is small in either case at late epochs. However, it is necessarily nonzero if the spatial coordinates are chosen to be comoving.

Now let us change coordinates to spatially noncomoving ones. Specifically, let us take new coordinates

$$T = tl \qquad R = t^{1/\alpha} r \qquad L = At^A l . \tag{4.26}$$

The 5-velocity transforms as a vector, $\overline{U}^A = \left(\partial \overline{x}^A / \partial x^B\right) U^B$, so the velocities in the new frame are

$$\overline{U}^0 = \mp \frac{\sqrt{2\alpha - 1}}{\alpha} \qquad \overline{U}^1 = \mp \frac{1}{\sqrt{2\alpha - 1}} \frac{R}{T} \qquad \overline{U}^4 = 0 , \tag{4.27}$$

along with $\bar{U}^2 = \bar{U}^3 = 0$. We see that the zeroth component is now a constant (this is related to energy in the usual interpretation), that the first component is finite (in fact Hubble's law), and that the fourth component is zero (defining a hypersurface $x^4 = l$ =constant). Since the metric transforms as a tensor, $\bar{g}^{AB} = \left(\partial \bar{x}^A / \partial x^C\right)\left(\partial \bar{x}^B / \partial x^D\right) g^{CD}$, we also find that in the new frame $\bar{g}^{00} = (2\alpha - 1)/\alpha^2 = 3/4$ ($\alpha = 2$) or 8/9 (α = 3/2). These results mean that in the new coordinates (4.26), there is a cosmic time which is the same for all observers, that galaxies move according to the usual law of expansion, and that energies and rest masses are constants (assuming we write $l = m$). In terms of the cosmic time T of equation (4.26), the density and pressure of the fluid are given by (4.13) as $8\pi\rho = 3/\alpha^2 T^2$ and $8\pi p = (2\alpha - 3)/\alpha^2 T^2$, which as we noted earlier are identical to the values for the early (α = 2) and late (α = 3/2) universe in the conventional 4D approach. In other words, it is the coordinates (4.26) rather than the coordinates in (4.12) which give back standard cosmology.

The physical interpretation of the algebra of the preceding paragraph is obvious. Cosmology in the coordinates of (4.12) has galaxies which remain at the same separations from each other, but the masses of all particles change slowly over times of order 10^{10} year at a rate given by (4.25). Cosmology in the coordinates of (4.26) has galaxies which distance themselves in accordance with Hubble's law from each other, and the masses of all particles are constant over time. In the latter frame, the microwave background and nucleosynthesis are due to the expansion of the universe. In the former frame, they are due to the variation of masses.

However, <u>both frames are valid</u>. This brings up an issue of more general scope. In 5D induced-matter theory, all coordinate transformations $x^A \to \bar{x}^A(x^B)$ are valid. But in 4D general relativity, only $\bar{x}^\alpha \to \bar{x}^\alpha(x^\beta)$ are considered, and they are a subset. This means that 5D

coordinate transformations can change 4D physics. [A particular example of this is that a coordinate transformation takes us from a universe with metric (4.2) that is empty to a universe with metric (4.12) that has matter.] For the case of cosmology, we have shown above that the conventional interpretation of observations implies a choice of coordinates where masses are constant, which of course agrees with the widespread view that they provide a system of standards (see Chapter 1). However, it is not really clear if this is the system of coordinates which is being used in astrophysics in practice; and the system with time-variable masses is actually very close. Thus consider (4.25) for the rate of change of the extra coordinate in the present (dust) universe. It predicts $\dot{m}/m \cong -0.11/t_0$ (where an overdot denotes the rate of change and t_0 is the present epoch). For $t_0 = 18 \times 10^9$ year we have $\dot{m}/m \cong -6 \times 10^{-12}\ yr^{-1}$. This is marginally consistent with ranging data from the Viking space probe to Mars (where the errors are reported as $\pm\ 4 \times 10^{-12}$ yr^{-1} and $\pm\ 10 \times 10^{-12}$ yr^{-1}) and quite consistent with timing data for the binary pulsar 1913 + 16 (where the errors are reported as $\pm\ 11 \times 10^{-12}$ yr^{-1}: see Will 1991). In the context of 5D theory, these data should be viewed as meaning that we are close to deciding by observation what system of coordinates is being used in astrophysics.

4.3 Spherically-Symmetric Astrophysical Systems

It is common practice to model astrophysical systems which are not grossly asymmetric as spherical clouds of matter with inhomogeneous density profiles. For example, rich or Abell clusters of galaxies can be modeled as static, isothermal spheres with power-law density profiles. In this section, we will study a general metric and then give a static solution with inverse-square isothermal matter (Billyard and Wesson 1996a). The latter is the 5D analog of a well-known 4D

solution (Henriksen and Wesson 1978). There are presumably many solutions of the 5D equations we will present below which contain known 4D solutions.

We take a spherically-symmetric metric in the form

$$dS^2 = e^{\nu}dt^2 - e^{\lambda}dr^2 - R^2 d\Omega^2 + \varepsilon e^{\mu}dl^2 \ . \tag{4.28}$$

Here the metric coefficients ν, λ, R, μ can depend in general on the time t, radius r and the extra coordinate l. The nonzero components of R_{AB} for (4.28) are

$$R_{00} = -\frac{\ddot{\lambda}}{2} - \frac{\ddot{\mu}}{2} - 2\frac{\ddot{R}}{R} + \frac{\dot{\nu}\dot{\lambda}}{4} + \frac{\dot{\nu}\dot{\mu}}{4} + \frac{\dot{\nu}\dot{R}}{R} - \frac{\dot{\lambda}^2}{4} - \frac{\dot{\mu}^2}{4}$$

$$+ e^{\nu-\lambda}\left(\frac{\nu''}{2} + \frac{\nu'^2}{4} - \frac{\nu'\lambda'}{4} + \frac{\nu'\mu'}{4} + \frac{\nu'R'}{R}\right) + \varepsilon e^{\nu-\mu}\left(-\frac{\overset{\bullet\bullet}{\nu}}{2} - \frac{\overset{\bullet}{\nu}^2}{4} + \frac{\dot{\nu}\overset{\bullet}{\mu}}{4} - \frac{\dot{\nu}\overset{\bullet}{\lambda}}{4} - \frac{\dot{\nu}\overset{\bullet}{R}}{R}\right)$$

$$R_{01} = -\frac{\dot{\mu}'}{2} - \frac{\dot{\mu}\mu'}{4} + \frac{\nu'\dot{\mu}}{4} + \frac{\dot{\lambda}\mu'}{4} + \frac{\dot{\lambda}R'}{R} + \frac{\nu'\dot{R}}{R} - \frac{2\dot{R}'}{R}$$

$$R_{04} = -\frac{\overset{\bullet}{\lambda}'}{2} - \frac{\dot{\lambda}\overset{\bullet}{\lambda}}{4} + \frac{\dot{\lambda}\overset{\bullet}{\nu}}{4} + \frac{\overset{\bullet}{\lambda}\dot{\mu}}{4} + \frac{\dot{\mu}\overset{\bullet}{R}}{R} + \frac{\overset{\bullet}{\nu}\dot{R}}{R} - \frac{2\overset{\bullet}{R}'}{R}$$

$$R_{11} = -\frac{\nu''}{2} - \frac{\mu''}{2} - \frac{\nu'^2}{4} - \frac{\mu'^2}{4} + \frac{\lambda'\nu'}{4} + \frac{\lambda'\mu'}{4} + \frac{\lambda'R'}{4}$$

$$- \frac{2R''}{R} + e^{\lambda-\nu}\left(\frac{\ddot{\lambda}}{2} + \frac{\dot{\lambda}^2}{4} - \frac{\dot{\lambda}\dot{\nu}}{4} + \frac{\dot{\lambda}\dot{\mu}}{4} + \frac{\dot{\lambda}\dot{R}}{R}\right) + \varepsilon e^{\lambda-\mu}\left(\frac{\overset{\bullet\bullet}{\lambda}}{2} + \frac{\overset{\bullet}{\lambda}^2}{4} - \frac{\dot{\lambda}\overset{\bullet}{\nu}}{4} - \frac{\dot{\lambda}\overset{\bullet}{\mu}}{4} + \frac{\dot{\lambda}\overset{\bullet}{R}}{R}\right)$$

$$R_{14} = -\frac{\overset{\bullet}{\nu}'}{2} - \frac{\nu'\overset{\bullet}{\nu}}{4} + \frac{\overset{\bullet}{\lambda}\nu'}{4} + \frac{\mu'\overset{\bullet}{\nu}}{4} + \frac{\overset{\bullet}{\lambda}R'}{R} + \frac{\mu'\overset{\bullet}{R}}{R} - \frac{2\overset{\bullet}{R}'}{R}$$

$$R_{22} = 1 + R^2 e^{-\nu}\left[\frac{\dot{R}^2}{R^2} + \frac{\ddot{R}}{R} - \frac{\dot{R}}{2R}(\dot{\nu} - \dot{\lambda} - \dot{\mu})\right]$$

$$- R^2 e^{-\lambda}\left[\frac{R'^2}{R^2} + \frac{R''}{R} + \frac{R'}{2R}(\nu' - \lambda' + \mu')\right] + \varepsilon R^2 e^{-\mu}\left[\frac{\overset{\bullet}{R}^2}{R^2} + \frac{\overset{\bullet\bullet}{R}}{R} + \frac{\overset{\bullet}{R}}{2R}(\overset{\bullet}{\nu} + \overset{\bullet}{\lambda} - \overset{\bullet}{\mu})\right]$$

$$R_{33} = (\sin^2\theta)R_{22}$$

$$R_{44} = -\frac{\ddot{\nu}}{2} - \frac{\dot{\nu}^2}{4} - \frac{\ddot{\lambda}}{2} - \frac{\dot{\lambda}^2}{4} + \frac{\dot{\mu}\dot{\nu}}{4} + \frac{\dot{\mu}\dot{\lambda}}{4} + \frac{\dot{\mu}\dot{R}}{R} - \frac{2\ddot{R}}{R}$$

$$-\varepsilon e^{\mu-\nu}\left(\frac{\ddot{\mu}}{2} + \frac{\dot{\mu}^2}{4} - \frac{\dot{\mu}\dot{\nu}}{4} + \frac{\dot{\mu}\dot{\lambda}}{4} + \frac{\dot{\mu}\dot{R}}{R}\right) + \varepsilon e^{\mu-\lambda}\left(\frac{\mu''}{2} + \frac{\mu'^2}{4} + \frac{\mu'\nu'}{4} - \frac{\mu'\lambda'}{4} + \frac{\mu'R'}{R}\right). \tag{4.29}$$

Here as before, a dot denotes $\partial/\partial t$, a prime denotes $\partial/\partial r$ and an asterisk denotes $\partial/\partial l$. The nonzero components of T_β^α for (4.28) are

$$8\pi T_0^0 = -e^{-\nu}\left(\frac{\dot{\lambda}\dot{\mu}}{4} + \frac{\dot{R}\dot{\mu}}{R}\right) + e^{-\lambda}\left(\frac{R'\mu'}{R} - \frac{\lambda'\mu'}{4} + \frac{\mu''}{2} + \frac{\mu'^2}{4}\right) - \varepsilon e^{-\mu}\left(\frac{\ddot{\lambda}}{2} + \frac{\dot{\lambda}^2}{4} - \frac{\dot{\mu}\dot{\lambda}}{4} + \frac{\dot{R}\dot{\lambda}}{R} - \frac{\dot{R}\dot{\mu}}{R} + \frac{\dot{R}^2}{R^2} + \frac{2\ddot{R}}{R}\right)$$

$$8\pi T_0^1 = -e^{-\lambda}\left(\frac{\dot{\mu}'}{2} + \frac{\dot{\mu}\mu'}{4} - \frac{\nu'\dot{\mu}}{4} - \frac{\dot{\lambda}\mu'}{4}\right)$$

$$8\pi T_1^1 = -e^{-\nu}\left(\frac{\ddot{\mu}}{2} + \frac{\dot{\mu}^2}{4} - \frac{\dot{\nu}\dot{\mu}}{4} + \frac{\dot{R}\dot{\mu}}{R}\right) + e^{-\lambda}\left(\frac{R'\mu'}{R} + \frac{\nu'\mu'}{4}\right) - \varepsilon e^{-\mu}\left(\frac{\dot{R}^2}{R^2} + \frac{2\ddot{R}}{R} + \frac{\dot{R}\dot{\nu}}{R} - \frac{\dot{R}\dot{\mu}}{R} + \frac{\ddot{\nu}}{2} + \frac{\dot{\nu}^2}{4} - \frac{\dot{\nu}\dot{\mu}}{4}\right)$$

$$8\pi T_2^2 = -e^{-\nu}\left(\frac{\dot{R}\dot{\mu}}{2R} - \frac{\dot{\nu}\dot{\mu}}{4} + \frac{\dot{\lambda}\dot{\mu}}{4} + \frac{\ddot{\mu}}{2} + \frac{\dot{\mu}^2}{4}\right) + e^{-\lambda}\left(\frac{R'\mu'}{2R} + \frac{\mu''}{2} + \frac{\mu'^2}{4} - \frac{\lambda'\mu'}{4} + \frac{\nu'\mu'}{4}\right)$$

$$-\varepsilon e^{-\mu}\left(\frac{\ddot{R}}{R} + \frac{\dot{R}\dot{\nu}}{2R} + \frac{\dot{R}\dot{\lambda}}{2R} - \frac{\dot{R}\dot{\mu}}{2R} + \frac{\ddot{\nu}}{2} + \frac{\dot{\nu}^2}{4} + \frac{\ddot{\lambda}}{2} + \frac{\dot{\lambda}^2}{4} + \frac{\dot{\nu}\dot{\lambda}}{4} - \frac{\dot{\nu}\dot{\mu}}{4} - \frac{\dot{\mu}\dot{\lambda}}{4}\right)$$

$$T_3^3 = T_2^2 \ . \tag{4.30}$$

We see that in general the fluid is anisotropic $(T_1^1 \neq T_2^2)$, and that in the time-dependent case there is a radial flow of energy $(T_0^1 \neq 0)$. However, it is apparent that previous relations simplify greatly for the time-independent case, to which we now turn.

A class of exact solutions to $R_{AB} = 0$ that is static is given by

$$dS^2 = \left(\frac{r}{r_0}\right)^{2(\alpha+1)} l^{2(\alpha+3)/\alpha} dt^2 - (3-\alpha^2) l^2 dr^2 - l^2 r^2 d\Omega^2 + 3(3\alpha^{-2}-1) r^2 dl^2 . \qquad (4.31)$$

Here r_0 is a constant and α is a parameter related to the properties of matter. The latter can be obtained by substituting (4.31) into (4.1) or (4.30) to give

$$8\pi T_0^0 = \frac{(2-\alpha^2)}{(3-\alpha^2) l^2 r^2}$$

$$8\pi T_1^1 = -\frac{(\alpha^2+2\alpha)}{(3-\alpha^2) l^2 r^2}$$

$$8\pi T_2^2 = -\frac{(\alpha^2+2\alpha+1)}{(3-\alpha^2) l^2 r^2}$$

$$T_3^3 = T_2^2 \quad . \qquad (4.32)$$

If as before we use $\bar{p} \equiv -(T_1^1 + T_2^2 + T_3^3)/3$ we have

$$8\pi \bar{p} = \frac{(\alpha^2+2\alpha+2/3)}{(3-\alpha^2) l^2 r^2} \qquad 8\pi\rho = \frac{(2-\alpha^2)}{(3-\alpha^2) l^2 r^2} \quad , \qquad (4.33)$$

so the equation of state is isothermal. A discussion of the physical properties of this solution has been given by Billyard and Wesson (1996a; the Kretschmann scalar for it is $K = 16/3r^4 l^4$). They have also solved the 5D geodesic equation (4.19) for a particle moving in the field of (4.31), and have pointed out that the latter can be applied to astrophysical problems such as the gravitational lensing of QSOs by clusters of galaxies.

4.4 Waves in a de Sitter Vacuum

The 4D de Sitter solution satisfies the Friedmann equations (4.8) with a 3D space that expands exponentially with time under the influence of the cosmological constant. The latter implies the equation of state $p = -\rho$ typical of the vacuum in general relativity (see Chapter 1).

This de Sitter vacuum has also been extensively used in quantum field theory, notably as a background for particle production. The 5D de Sitter solution is one of the models that reduce on hypersurfaces x^4 = constants to FRW ones, as found by Ponce de Leon (1988). However, his solution may be transformed by a complex coordinate transformation into a metric that has wave-like properties, as found by Billyard and Wesson (1996b). The existence of wave-like solutions of the field equations $R_{AB} = 0$ raises major questions of interpretation, to do with the meaning of complex metric coefficients and the nature of the induced matter associated with the waves. We will return to these issues later; but for now we wish to present a wave-like solution and give a physical interpretation of it.

Consider then the x^4-dependent 5D metric specified by

$$dS^2 = l^2dt^2 - l^2e^{i(\omega t+k_x x)}dx^2 - l^2e^{i(\omega t+k_y y)}dy^2 - l^2e^{i(\omega t+k_z z)}dz^2 + L^2dl^2 \,. \tag{4.34}$$

Here ω is a frequency, k_x etc. are wavenumbers, and L measures the size of the extra dimension. The nonzero components of the Riemann-Christoffel tensor for (4.34) have the form

$$R_{txtx} = -\left(\frac{l^2}{4L^2}\right)\left(\omega^2L^2 - 4\right)e^{i(\omega t+k_x x)} \quad etc.$$

$$R_{xyxy} = \left(\frac{l^2}{4L^2}\right)\left(\omega^2L^2 - 4\right)e^{i(2\omega t+k_x x+k_y y)} \; etc., \tag{4.35}$$

with two similar relations in each case obtained by permutations of the space coordinates. The nonzero components of the Ricci tensor look like

$$R_{tt} = -\left(\frac{3}{4L^2}\right)\left(\omega^2L^2 - 4\right)$$

$$R_{xx} = \left(\frac{3}{4L^2}\right)\left(\omega^2L^2 - 4\right)e^{i(\omega t+k_x x)} \quad etc. \tag{4.36}$$

It is clear from these relations that the field equations $R_{AB} = 0$ are satisfied for $\omega = \pm 2/L$. It is also clear from (4.35) that the solution is 5D flat.

However, it is not flat in 4D and implies non-trivial physics in the induced-matter picture. Thus (2.52) gives the 4D Ricci or curvature scalar as

$$R = \frac{1}{4L^2}\left[g^{*\mu\nu}\, g^{\bullet}_{\mu\nu} + \left(g^{\mu\nu}\, g^{\bullet}_{\mu\nu} \right)^2 \right] = \frac{12}{l^2 L^2} \,. \tag{4.37}$$

And (2.53) or equivalently (4.1) gives the 4D effective or induced energy-momentum tensor as

$$8\pi T_{00} = -\frac{3\omega^2}{4}$$

$$8\pi T_{11} = \frac{3\omega^2}{4} e^{i(\omega t + k_x x)} \quad etc. \tag{4.38}$$

It is possible, as we have done in other cases, to match these components to those of a perfect fluid with $T_{\alpha\beta} = (\rho + p)u_\alpha u_\beta - p g_{\alpha\beta}$. This requires knowledge of the 4-velocities u_α, which can be obtained by solving the 5D geodesic equation (4.19). However, the working is tedious, so we only quote the result (see Billyard and Wesson 1996b). This is that the induced matter for metric (4.34) has a pressure and density given by

$$8\pi p = -8\pi\rho = \frac{3\omega^2}{4l^2}. \tag{4.39}$$

That is, the equation of state is that of the vacuum in general relativity. In fact, the induced medium is a de Sitter vacuum which if interpreted in terms of the cosmological constant would have $\Lambda = -3\omega^2 / 4l^2$.

The 5D solution (4.34) is curious but highly instructive. Its 3D part has wave-like metric coefficients which are complex, but there is no problem with interpreting real and imaginary parts of the geometric quantities associated with it, and the physical quantities associated with it are all real. The 'vacuum waves' in (4.34) exist, of course, because of the choice of coordinates.

We can if we wish absorb the oscillatory terms in 3D by appropriate coordinate transformations, and ω can be made imaginary (which reverses the sign of the last part of the metric since $L^2 = 4/\omega^2$). The result is a 5D growing-mode de Sitter solution of the type originally found by Ponce de Leon (1988). It should, however, be noted that the standard 4D growing-mode de Sitter solution can if one wishes also be converted to an oscillatory mode. The easiest way to see that even 4D relativity admits vacuum waves is to look at the Friedman equations (4.8). These are satisfied for $\rho_m = p_m = 0$, k = 0, $S = e^{i\omega t}$ and $\Lambda = -3\omega^2$. Previously, such 4D oscillatory motion would have been viewed as a mathematical quirk, but it is given legitimacy by the 5D induced-matter approach. In either 5D or 4D, solutions of the kind we have been discussing may be relevant to the early universe. For we can imagine a static model full of vacuum waves with real ω, which undergoes a phase change to imaginary ω, resulting in an expanding model of the inflationary type.

4.5 Time-Dependent Solitons

The solitons studied in Section 2.4 are independent of $x^0 = t$ and $x^4 = l$, but in the absence of a Birkhoff-type theorem they can be generalized in both regards (see Liu, Wesson and Ponce de Leon 1993, and Ponce de Leon and Wesson 1993, respectively). The time-dependent extension is particularly interesting, because if static solitons exist in nature they presumably formed by some dynamical process. Also, time-dependent solutions which represent centrally-condensed clouds of hot matter may be applicable to the problem of galaxy formation. Thus we will give a class of time-dependent soliton solutions (Wesson, Liu and Lim 1993) and evaluate their induced properties of matter.

Let us consider the time-dependent generalization of the static soliton metric (2.23) in the form

$$dS^2 = A^2(t,r)dt^2 - B^2(t,r)dx^i dx^i - C^2(t,r)dl^2. \tag{4.40}$$

We are here for algebraic convenience using Cartesian coordinates for the 3-space, which are related to spherical polar coordinates by the usual relations ($x^1 = r\ sin\ \theta\ cos\ \phi$, $x^2 = r\ sin\theta\ sin\ \phi$, $x^3 = r\ cos\ \theta$). For the metric (4.40), the nonzero components of the 5D Ricci tensor are

$$R_{00} = \frac{AA'}{B^2}\left(\frac{2}{r}+\frac{A''}{A'}+\frac{B'}{B}+\frac{C'}{C}\right)+\frac{\dot{A}}{A}\left(\frac{3\dot{B}}{B}+\frac{\dot{C}}{C}\right)-\left(\frac{3\ddot{B}}{B}+\frac{\ddot{C}}{C}\right)$$

$$R_{0i} = \left(\frac{x^i}{r}\right)\left[\frac{\dot{B}}{B}\left(\frac{2A'}{A}+\frac{2B'}{B}+\frac{C'}{C}\right)+\frac{A'\dot{C}}{AC}-\frac{2\dot{B}'}{B}-\frac{\dot{C}'}{C}\right]$$

$$R_{ij} = -\delta_{ij}\left[-\frac{B\dot{B}}{A^2}\left(\frac{\ddot{B}}{\dot{B}}+\frac{2\dot{B}}{B}-\frac{\dot{A}}{A}+\frac{\dot{C}}{C}\right)+\frac{1}{r}\left(\frac{A'}{A}+\frac{3B'}{B}+\frac{C'}{C}\right)+\frac{B''}{B}+\frac{B'}{B}\left(\frac{A'}{A}+\frac{C'}{C}\right)\right]$$

$$-\frac{x^i x^j}{r^2}\left[\frac{A''}{A}+\frac{B''}{B}+\frac{C''}{C}-\left(\frac{1}{r}+\frac{2B'}{B}\right)\left(\frac{A'}{A}+\frac{B'}{B}+\frac{C'}{C}\right)\right]$$

$$R_{44} = -\frac{CC'}{B^2}\left(\frac{2}{r}+\frac{A'}{A}+\frac{B'}{B}+\frac{C''}{C'}\right)+\frac{C\dot{C}}{A^2}\left(-\frac{\dot{A}}{A}+\frac{3\dot{B}}{B}+\frac{\ddot{C}}{C}\right). \tag{4.41}$$

Using these, we look for solutions of $R_{AB} = 0$ that are separable:

$$A = \alpha(t)a(r) \qquad B = \beta(t)b(r) \qquad C = \gamma(t)c(r)\ . \tag{4.42}$$

There are now 6 functions to be determined, but from (4.40) we see that a coordinate transformation on t allows us to set $\alpha(t) = 1$ without loss of generality. Also, let us suppose that $a(r)$, $b(r)$, $c(r)$ are the same functions as in the static case. [Here $a(r)$ should not be confused with the constant a of the static soliton.] Then the field equations with (4.41) give 4 relations:

$$0 = \frac{3\ddot{\beta}}{\beta}+\frac{\ddot{\gamma}}{\gamma}$$

$$0=(2k-1)\frac{\dot{\beta}}{\beta}+(\kappa+1)\frac{\dot{\gamma}}{\gamma}$$

$$0=\frac{\ddot{\beta}}{\beta}+\frac{\dot{\beta}}{\beta}\left(\frac{2\dot{\beta}}{\beta}+\frac{\dot{\gamma}}{\gamma}\right)$$

$$0=\frac{\ddot{\gamma}}{\gamma}+\frac{3\dot{\beta}\dot{\gamma}}{\beta\gamma} \quad . \tag{4.43}$$

Here κ is the constant that appears in the static soliton metric and figures in the consistency relation $\varepsilon^2(\kappa^2-\kappa+1)=1$ of (2.25). It turns out to be fixed by the solution of (4.43), which is

$$\beta=\gamma^{-1}=(1+Ht)^{1/2} \qquad k=2 \qquad \varepsilon=1/\sqrt{3} \quad . \tag{4.44}$$

Here H is a constant with the physical dimensions of an inverse time and in a cosmological application would therefore be analogous to Hubble's constant. Putting previous relations back into (4.40) makes the latter read

$$dS^2=\left(\frac{ar-1}{ar+1}\right)^{4/\sqrt{3}}dt^2-\left(\frac{a^2r^2-1}{a^2r^2}\right)^2\left(\frac{ar+1}{ar-1}\right)^{2/\sqrt{3}}(1+Ht)dx^i dx^i-\left(\frac{ar+1}{ar-1}\right)^{2/\sqrt{3}}(1+Ht)^{-1}dl^2 \tag{4.45}$$

This describes a time-dependent soliton, as opposed to the static one (2.24), and $a = 2/M_*$ defines the dimensional constant in terms of a mass. We see that for $a\to\infty$ ($M_*\to 0$) the metric (4.45) tends to the RW one (2.8), which describes a cosmology where the matter consists of photons.

The matter associated with (4.45) in the general case can be obtained from (4.1). In mixed form, the components of the induced energy-momentum tensor are

$$8\pi T_0^0=\frac{1}{(1+Ht)}\,\frac{8a^6r^4}{3\left(a^2r^2-1\right)^4}\left(\frac{ar-1}{ar+1}\right)^{2/\sqrt{3}}+\frac{3H^2}{4(1+Ht)^2}\left(\frac{ar-1}{ar+1}\right)^{-4/\sqrt{3}}$$

$$8\pi T_0^1 = -\frac{H}{(1+Ht)^2}\frac{4a}{\sqrt{3}}\frac{a^4r^4}{\left(a^2r^2-1\right)^3}\left(\frac{ar-1}{ar+1}\right)^{2/\sqrt{3}}$$

$$8\pi T_1^1 = -\frac{1}{(1+Ht)}\frac{4a^5r^3}{\sqrt{3}\left(a^2r^2-1\right)^4}\left(\frac{ar-1}{ar+1}\right)^{2/\sqrt{3}}\left(a^2r^2+1\right)-\frac{H^2}{4(1+Ht)^2}\left(\frac{ar-1}{ar+1}\right)^{-4/\sqrt{3}}$$

$$8\pi T_2^2 = -\frac{1}{(1+Ht)}\frac{2a^5r^3}{\sqrt{3}\left(a^2r^2-1\right)^4}\left(\frac{ar-1}{ar+1}\right)^{2/\sqrt{3}}\left(\frac{2}{\sqrt{3}}ar-a^2r^2-1\right)-\frac{H^2}{4(1+Ht)^2}\left(\frac{ar-1}{ar+1}\right)^{-4/\sqrt{3}}$$

$$T_3^3 = T_2^2 \quad . \tag{4.46}$$

These can be compared to those for the static soliton solution (4.15) and the static spherically-symmetric solution (4.32). We see that the fluid is anisotropic with the equation of state $T_\alpha^\alpha = 0$, which in terms of an averaged 3D pressure $\bar{p}$ is

$$\rho = 3\bar{p} = \frac{1}{(1+Ht)}\frac{8a^6r^4}{3\left(a^2r^2-1\right)^4}\left(\frac{ar-1}{ar+1}\right)^{2/\sqrt{3}}+\frac{3H^2}{4(1+Ht)^2}\left(\frac{ar+1}{ar-1}\right)^{4/\sqrt{3}} \quad . \tag{4.47}$$

The first term of this expression is similar to the static case but multiplied by a time factor. The second term represents a contribution to the density and pressure from the motion. Thus we have a natural generalization of the properties of matter of the static soliton.

4.6 Systems with Axial and Cylindrical Symmetry

In Section 4.3 we studied spherically-symmetric solutions because they are relevant to astrophysics. However, systems with axial and cylindrical symmetry are also relevant, notably in 2 ways. First, axially-symmetric solutions can describe objects with spin. In this connection, we note that there has been considerable work on 5D Kerr-like solutions (Gross and Perry 1983; Myers and Perry 1986; Clement 1986; Bruckman 1986, 1987; Frolov, Zelnikov and Bleyer 1987; Horne and Horowitz 1992; Matos 1994). However, some of these solutions are contrived and none involves x^4-dependence. Introducing the latter is difficult, and attempts to do it both

algebraically and by computer have proven fruitless. This means that the vintage problem of finding a realistic matter source for the external 4D Kerr metric (which is a natural one for the induced-matter approach) remains unsolved. Second, cylindrically-symmetric solutions offer new insight on the question of the inevitability of cosmological singularities. Perhaps unexpectedly, exact solutions of the 4D field equations have been found which while somewhat unphysical have extended matter of cosmological type but no singularities (Senovilla 1990; Chinea, Fernandez-Jambrina and Senovilla 1992; Ruiz and Senovilla 1992; Tikekar, Patel and Dadhich 1994). The interest caused by these 4D solutions has led to their generalization to 5D (Banerjee, Das and Panigrahi 1995; Chatterjee, Wesson and Billyard 1997). We proceed to a brief account of these.

Consider the metric

$$dS^2 = A^2\left(dt^2 - dr^2\right) - B^2\,dy^2 - C^2dz^2 - D^2dl^2\ , \tag{4.48}$$

where A-D depend only on t,r. As we know, this condition implies an equation of state $T^{\alpha}_{\alpha} = 0$, where by (4.1) the components of the energy-momentum tensor are:

$$8\pi T^0_0 = \frac{1}{A^2}\left[\frac{\ddot{D}}{D} - \frac{\dot{A}\dot{D}}{AD} - \frac{A'D'}{AD}\right]$$

$$8\pi T^1_0 = -\frac{1}{A^2}\left[\frac{\dot{D}'}{D} - \frac{A'\dot{D}}{AD} - \frac{\dot{A}D'}{AD}\right]$$

$$8\pi T^1_1 = -\frac{1}{A^2}\left[\frac{D''}{D} - \frac{\dot{A}\dot{D}}{AD} - \frac{A'D'}{AD}\right]$$

$$8\pi T^2_2 = -\frac{1}{A^2}\left[-\frac{\dot{B}\dot{D}}{BD} + \frac{B'D'}{BD}\right]$$

$$8\pi T^3_3 = -\frac{1}{A^2}\left[-\frac{\dot{C}\dot{D}}{CD} + \frac{C'D'}{CD}\right]\ . \tag{4.49}$$

These can be evaluated given a solution for *A-D* of $R_{AB} = 0$, and 4 cases have been studied in the induced-matter picture (Chatterjee, Wesson and Billyard 1997). Their properties are similar, however, so we quote just one:

$$A = \cosh(2qt) \cdot \cosh^4(qr)$$

$$B = q^{-1} \cosh(2qt) \cdot \cosh(qr) \cdot \sinh(qr)$$

$$C = \cosh(2qt) \cdot \cosh^2(qr)$$

$$D = [\cosh(2qt)]^{-1} [\cosh(qr)]^{-2}$$

$$8\pi T_0^0 = 4q^2 [\cosh(2qt)]^{-2} [\cosh(qr)]^{-8} [3\tanh^2(2qt) + 2\tanh^2(qr) - 1]$$

$$8\pi T_0^1 = -16q^2 \sinh(2qt) [\cosh(2qt)]^{-3} \sinh(qr) [\cosh(qr)]^{-9}$$

$$8\pi T_1^1 = -2q^2 [\cosh(2qt)]^{-2} [\cosh(qr)]^{-8} [2\tanh^2(2qt) + 7\tanh^2(qr) - 1]$$

$$8\pi T_2^2 = -2q^2 [\cosh(2qt)]^{-2} [\cosh(qr)]^{-8} [2\tanh^2(2qt) - \tanh^2(qr) - 1]$$

$$8\pi T_3^3 = -4q^2 [\cosh(2qt)]^{-2} [\cosh(qr)]^{-8} [\tanh^2(2qt) - \tanh^2(qr)]$$

$$K = 192q^4 [\cosh(qr)]^{-20} [\cosh(2qt)]^{-8} [8\cosh^2(2qt)\sinh^2(2qt)\cosh^4(qr)$$

$$+ 6\cosh^4(qr) - 14\cosh^4(2qt)\cosh^2(qr) + 9\cosh^4(2qt)]. \qquad (4.50)$$

Here q is a constant with the physical dimension of an inverse length, and the Kretschmann scalar is 192 q^4 at the origin of space and time.

The same result holds for all members of this class, and should be compared to that of more familiar cosmological models. For example, the 5D radiation model (2.8) has $K = 9/2t^4$, confirming that there is a big-bang singularity at $t = 0$. This behaviour is similar to that of the Kasner-like 5D solutions studied by Roque and Seiler (1991). They found several classes of

solutions that generalize to the anisotropic case the isotropic 5D solutions of Ponce de Leon (1988). The solutions of Roque and Seiler are Bianchi type I on the hypersurfaces l = constants, but in general the metric coefficients of the 3 spatial dimensions and the extra dimension can depend on t and l, so in general these models do not have radiation-like matter. As for singularities, Roque and Seiler noted the existence of non-flat solutions whose Kretchmann scalars diverged for $t \to 0$ and $l \to 0$, and found a non-flat solution with zero K. They also found 5D flat solutions with (of course) zero K. The Ponce de Leon solution (4.12) is of the latter type, and while in this section we have looked at some unusual models, we should remind ourselves that the standard cosmology is 5D flat and has no geometrical singularity.

4.7 Shell-Like and Flat Systems

In Section 4.4 we mentioned the inflationary-universe model, and irrespective of whether there was a big-bang singularity it is widely believed that the presently observed structure of galaxies, clusters and superclusters is due to processes that happened in an early vacuum-dominated period. These processes could have included phase changes and the formation of bubbles and other particle-like objects, and are essentially quantum in nature. However, we need corresponding descriptions in general relativity, especially for later epochs when the universe became classical in nature. Let us therefore examine shell-like systems in induced-matter theory.

Specifically, we take a metric of the form called canonical (Mashhoon, Liu and Wesson 1994) and give a typical class of shell-like solutions (Wesson and Liu 1998). We will study general implications of the canonical metric later, but for now we consider a metric of the form

$$dS^2 = \frac{l^2}{L^2}\left[A^2dt^2 - B^2dr^2 - C^2r^2d\Omega^2\right] - dl^2 \ . \tag{4.51}$$

Here L is a constant introduced for dimensional consistency, and the metric coefficients A, B, C depend on r only. For (4.51), the nonzero components of the 5D Ricci tensor are:

$$R_{00} = \frac{A^2}{B^2}\left\{\left[\frac{A''}{A} + \frac{A'}{A}\left(-\frac{B'}{B} + \frac{2C'}{C} + \frac{2}{r}\right) + \frac{3B^2}{L^2}\right]\right.$$

$$\left. + \frac{l^2B^2}{L^2}\left[\frac{\ddot{A}}{A} + \frac{1}{l}\left(\frac{5\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{2\dot{C}}{C}\right) + \frac{\dot{A}}{A}\left(\frac{\dot{B}}{B} + \frac{2\dot{C}}{C}\right)\right]\right\}$$

$$R_{11} = -\frac{A''}{A} - \frac{2C''}{C} + \frac{B'}{B}\left(\frac{A'}{A} + \frac{2C'}{C}\right) + \frac{2}{r}\left(\frac{B'}{B} - \frac{2C'}{C}\right) - \frac{3B^2}{L^2}$$

$$- \frac{l^2B^2}{L^2}\left[\frac{\ddot{B}}{B} + \frac{1}{l}\left(\frac{\dot{A}}{A} + \frac{5\dot{B}}{B} + \frac{2\dot{C}}{C}\right) + \frac{\dot{B}}{B}\left(\frac{\dot{A}}{A} + \frac{2\dot{C}}{C}\right)\right]$$

$$R_{22} = -r^2\frac{C^2}{B^2}\left\{\left[\frac{C''}{C} + \frac{C'}{C}\left(\frac{A'}{A} - \frac{B'}{B} + \frac{C'}{C}\right) + \frac{1}{r}\left(\frac{A'}{A} - \frac{B'}{B} + \frac{4C'}{C}\right) + \frac{1}{r^2}\left(1 - \frac{B^2}{C^2}\right) + \frac{3B^2}{L^2}\right]\right.$$

$$\left. + \frac{l^2B^2}{L^2}\left[\frac{\ddot{C}}{C} + \frac{1}{l}\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{6\dot{C}}{C}\right) + \frac{\dot{C}}{C}\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right)\right]\right\}$$

$$R_{14} = -\frac{\dot{A}'}{A} - \frac{2\dot{C}'}{C} + \frac{\dot{B}}{B}\left(\frac{A'}{A} + \frac{2C'}{C}\right) + \frac{2}{r}\left(\frac{\dot{B}}{B} - \frac{\dot{C}}{C}\right)$$

$$R_{44} = -\left(\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{2\ddot{C}}{C}\right) - \frac{2}{l}\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{2\dot{C}}{C}\right) \quad . \tag{4.52}$$

Here a prime denotes $\partial / \partial r$ and an asterisk denotes $\partial / \partial l$ as elsewhere. It may be verified that the components (4.52) are zero for:

$$A = k_1\left(1 - \frac{r^2}{L^2}\right)^{1/2} + \frac{k_2 L}{l}$$

$$B = \frac{1}{\left(1 - r^2 / L^2\right)^{1/2}}$$

$$C = 1 + \frac{k_3 L^2}{rl} \,. \tag{4.53}$$

Here k_1, k_2 and k_3 are arbitrary dimensionless constants. The other constant L is a length in the case where the coordinates r and l are lengths, but in principle it can be suppressed using an appropriate choice of units via $L = 1$. The same cannot be done for k_1, k_2 and k_3, so (4.53) is a 3-parameter class.

The properties of matter associated with (4.53) can be evaluated using (4.1). However, for metrics of canonical form there is a more convenient formalism (Mashhoon, Liu and Wesson 1994). In terms of the 4D metric tensor $g_{\mu\nu}$ and its associated quantity $A_{\mu\nu} \equiv \dot{g}_{\mu\nu}/2$, the induced energy-momentum tensor is

$$8\pi T_{\mu\nu} = \frac{3}{L^2} g_{\mu\nu} - \frac{l^2}{L^2}\left[\dot{A}_{\mu\nu} + \left(\frac{4}{l} + A^{\alpha}_{\alpha}\right)\left(A_{\mu\nu} - \frac{1}{2} g_{\mu\nu} A^{\beta}_{\beta}\right) - 2\left(A^{\alpha}_{\mu} A_{\nu\alpha} - \frac{1}{4} g_{\mu\nu} A^{\alpha\beta} A_{\alpha\beta}\right)\right]. \tag{4.54}$$

For metric (4.51), we have $A^{\alpha}_{\alpha} = \dot{A}/A + 2\dot{C}/C$ and $A^{\alpha\beta} A_{\alpha\beta} = \left(\dot{A}/A\right)^2 + 2\left(\dot{C}/C\right)^2$. It is also useful to define

$$F \equiv 1 - \frac{r^2}{L^2}$$

$$H \equiv \frac{3}{L^2} + \frac{l^2}{L^2}\left[\frac{\dot{C}^2}{C^2} + \frac{2\dot{A}\dot{C}}{AC} + \frac{2}{l}\left(\frac{\dot{A}}{A} + \frac{2\dot{C}}{C}\right)\right] = \frac{1}{L^2}\left(\frac{1}{C^2} + \frac{2k_1\sqrt{F}}{AC}\right)$$

$$N \equiv \frac{\dot{A}}{A} + \frac{2\dot{C}}{C} + \frac{4}{l} = \frac{1}{l}\left(\frac{k_1\sqrt{F}}{A} + \frac{2}{C} + 1\right). \tag{4.55}$$

Then (4.54) becomes

$$8\pi T_{\mu\nu} = Hg_{\mu\nu} - \frac{l^2}{2L^2}\left[\overset{**}{g}_{\mu\nu} + N\overset{*}{g}_{\mu\nu} - g^{\alpha\beta}\overset{*}{g}_{\mu\alpha}\overset{*}{g}_{\nu\beta}\right]. \tag{4.56}$$

Evaluating this and expressing it in mixed form gives

$$8\pi T_0^0 = \frac{1}{L^2}\left(\frac{2}{C} + \frac{1}{C^2}\right)$$

$$8\pi T_1^1 = \frac{1}{L^2}\left(\frac{2k_1\sqrt{F}}{AC} + \frac{1}{C^2}\right)$$

$$8\pi T_2^2 = \frac{1}{L^2}\left(\frac{k_1\sqrt{F}}{A} + \frac{k_1\sqrt{F}}{AC} + \frac{1}{C}\right).$$

$$T_3^3 = T_2^2 \quad . \tag{4.57}$$

These describe a fluid which like that of a soliton is static, spherically-symmetric and anisotropic $\left(T_1^1 \neq T_2^2\right)$. But unlike that object, (4.54) has matter concentrated on shells: the density diverges where $C = 0$ and the pressure diverges where $A = 0$ and $C = 0$. These latter conditions, by (4.53), occur respectively on the surfaces

$$r_A = L\left[1 - \left(\frac{k_2}{k_1}\right)^2\left(\frac{L}{l}\right)^2\right]^{1/2}$$

$$r_C = \frac{|k_3|L^2}{l} \quad . \tag{4.58}$$

These are the radii of surfaces where the matter properties are formally singular in a distribution which is otherwise regular (we have assumed l, $L > 0$, $k_1k_2 < 0$ and $k_3 < 0$). The physics of the matter described by (4.57), (4.58) could be discussed at length, but we restrict ourselves to the following comments. (a) The surface at r_A can be interpreted as the edge of a bubble during the phase change associated with inflationary universe models; or as the surface where the stresses reside necessary to stabilize classical models of elementary particles. (b) If desired, the surface

at r_C can be redefined as the centre of the distribution, since by (4.51) the surface area of 2D shells goes to zero as $C \to 0$. But this is not necessary, and can also be avoided by simply setting $k_3 = 0$ so $C = 1$ in (4.53), to give a 2-parameter class of single-shell solutions. (c) The divergence in the matter properties could be inferred from an inspection of the metric (4.51) and the solution (4.53), but its geometrical origin can be better appreciated by forming the 4D Ricci scalar. This is related to the trace of the energy-momentum tensor via

$$R = -8\pi T^{\alpha}_{\alpha} = -\frac{2}{L^2}\left(\frac{k_1\sqrt{F}}{A} + \frac{2k_1\sqrt{F}}{AC} + \frac{1}{C^2} + \frac{2}{C}\right) \quad , \tag{4.59}$$

and is obviously singular at $A = 0$, $C = 0$. (d) The matter is concentrated on shells because of the nature of the 4D as opposed to the 5D geometry. Indeed, the solution (4.53) is not only Ricci-flat $\left(R_{AB} = 0\right)$ but may also be shown to be Riemann-flat $\left(R_{ABCD} = 0\right)$. In other words, the properties of the matter are due to the fact that a curved 4D space is embedded in a flat 5D space.

Let us then look at solutions which are curved in 4D but flat in 5D, concentrating not on the cosmological case with which we are already familiar but on the more general spherically-symmetric case. Such solutions are not trivial. While one can easily write down $R_{AB} = 0$ and $R_{ABCD} = 0$ in 5D, considerable skill is required in choosing a metric that gives back physically meaningful results from $G_{\alpha\beta} = 8\pi T_{\alpha\beta}$ in 4D. Presently, there is no rule known which shows how to go from a flat 5D space to a curved 4D one and have the latter possess reasonable physics. However, Abolghasem, Coley and McManus (1996) used a metric in (3D) isotropic form and looked for solutions which admit the transformations (diffeomorphisms) $x^{\alpha} \to \bar{x}^{\alpha}(x^{\beta})$, $x^4 \to \bar{x}^4(x^4)$, and found a class of solutions whose induced energy-momentum tensor is that of a perfect fluid and so may be applied to astrophysics. Also, Liu and Wesson (1998) extended their work using a metric in standard (non-isotropic) coordinates, and found a

broad class of solutions which admits a more general (anisotropic) energy-momentum tensor and may be applied to several areas of physics. We proceed to give a brief account of the latter class of solutions in order to illustrate the utility of 5D flatness.

Consider a metric which has significant dependence on the extra coordinate but whose 4D part preserves its signature under real transformations of all 5 coordinates:

$$dS^2 = A^2dt^2 - B^2dr^2 - C^2d\Omega^2 - dl^2$$

$$A \equiv A_1 + lA_2 \qquad B \equiv B_1 + lB_2 \qquad C \equiv C_1 + lC_2. \tag{4.60}$$

Here A_1, A_2, B_1, B_2, C_1, C_2 are functions of r. We have solved $R_{AB} = 0$ for (4.60) algebraically, but do not write out the components of this because the more numerous equation $R_{ABCD} = 0$ may be confirmed by computer using programs such as GRTensor (Lake, Musgrave and Pollney 1995). Thus a class of solutions is given by

$$A_1' = \frac{-aC_2C_1'}{\left(1-C_2^2\right)^{1/2}} \qquad A_2 = a\left(1-C_2^2\right)^{1/2}$$

$$B_1 = \frac{C_1'}{\left(1-C_2^2\right)^{1/2}} \qquad B_2 = \frac{C_2'}{\left(1-C_2^2\right)^{1/2}}\,. \tag{4.61}$$

Here a is an arbitrary constant, a prime denotes $\partial / \partial r$ and $C_1(r)$, $C_2(r)$ remain arbitrary. Of these latter two functions, one can in principle be absorbed by an appropriate choice of the r coordinate. However, while this is mathematically possible, it is often more instructive to choose C_1 and C_2 so as to give the metric and its corresponding induced matter relevance to some physical situation.

For example, if we choose $C_2 = r/L$ where L is some length and put $a = 1/L$, we obtain

$$dS^2 = \frac{l^2}{L^2}\left\{\left[\left(1-\frac{r^2}{L^2}\right)^{1/2} + \frac{L}{l}A_1(r)\right]^2 dt^2 - \frac{\left[1+LC_1'/l\right]^2}{\left[1-r^2/L^2\right]}dr^2 - r^2\left[1+\frac{L}{l}\frac{C_1}{r}\right]^2 d\Omega^2\right\} - dl^2$$

$$A_1 = A_0 - \frac{1}{L^2}\int \frac{rC_1' dr}{\left(1-r^2/L^2\right)^{1/2}} \,. \tag{4.62}$$

Here A_0 is an arbitrary constant and $C_1(r)$ is still an arbitrary function. We recognize this as a generalization of the 5D solution of de Sitter type

$$dS^2 = \frac{l^2}{L^2}\left\{\left[1-\frac{r^2}{L^2}\right]dt^2 - \frac{dr^2}{\left[1-r^2/L^2\right]} - r^2 d\Omega^2\right\} - dl^2 \,, \tag{4.63}$$

which we recover if $C_1 = 0$ and $A_0 = 0$. If alternatively $C_1 = 0$ but $A_0 \neq 0$ then

$$dS^2 = \frac{l^2}{L^2}\left\{\left[\left(1-\frac{r^2}{L^2}\right)^{1/2} + \frac{A_0 L}{l}\right]^2 dt^2 - \frac{dr^2}{\left(1-r^2/L^2\right)} - r^2 d\Omega^2\right\} - dl^2 \,. \tag{4.64}$$

And if $C_1 = C_0 =$ constant, (4.62) gives

$$dS^2 = \frac{l^2}{L^2}\left\{\left[\left(1-\frac{r^2}{L^2}\right)^{1/2} + \frac{A_0 L}{l}\right]^2 dt^2 - \frac{dr^2}{\left(1-r^2/L^2\right)} - r^2\left(1+\frac{C_0 L}{lr}\right)^2 d\Omega^2\right\} - dl^2 \,. \tag{4.65}$$

An inspection of (4.64) and (4.65) shows that the physics they imply can be quite different from that of the conventional solution (4.63). Since the 4D de Sitter solution contained in (4.63) is the spacetime often regarded in particle physics as defining the vacuum, one can argue that the 5D solution (4.62) represents a 'generalized' vacuum.

As another example, if we choose $C_1 = r$ in (4.61) we obtain

$$dS^2 = \left(A_1 + lA_2\right)^2 dt^2 - \left(B_1 + lB_2\right)^2 dr^2 - \left(r + lC_2\right)^2 d\Omega^2 - dl^2$$

$$A_1 = A_0 - a\int \frac{C_2 dr}{\left(1-C_2^2\right)^{1/2}} \qquad A_2 = a\left(1-C_2^2\right)^{1/2}$$

$$B_1 = \frac{1}{\left(1-C_2^2\right)^{1/2}} \qquad B_2 = \frac{C_2'}{\left(1-C_2^2\right)^{1/2}} \,. \tag{4.66}$$

Here a, A_0 are arbitrary constants and $C_2(r)$ is arbitrary. The last can, however, be replaced by any other of A_1, A_2, B_1, B_2 (i.e. we can regard C_2 as being determined by one of these other functions). Let us choose to keep $A_1(r)$ arbitrary and determine C_2 from $C_2 = \left[1+\left(a / A_1'\right)^2\right]^{-1/2}$ in accordance with (4.66). Rewriting $A_1(r)$ in terms of a constant M and a new arbitrary function $h(r)$ we have

$$A_1(r) = \left[1 - \frac{2M}{r} + 2h(r)\right]^{1/2}$$

$$C_2(r) = \frac{1}{a}\left[\frac{M}{r^2} + h'\right]\left[1 - \frac{2M}{r} + 2h + \frac{1}{a^2}\left(\frac{M}{r^2} + h'\right)^2\right]^{-1/2} . \tag{4.67}$$

If $h(r) \rightarrow 0$, we find that

$$dS^2 \cong \left[\left(1 - \frac{2M}{r}\right)^{1/2} + al\right]^2 dt^2 - dr^2 - r^2 d\Omega^2 - dl^2 . \tag{4.68}$$

This will give Newtonian physics for $a \rightarrow 0$ and the usual weak-field limit $(2M/r << 1)$. But it does not reproduce the Schwarzschild solution

$$dS^2 \cong \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 + \frac{2M}{r}\right) dr^2 - r^2 d\Omega^2 - dl^2 . \tag{4.69}$$

This is as expected, since as mentioned before the exact 4D Schwarzschild solution can only be embedded in a flat manifold if the latter has 6 or more dimensions. However, (4.68) is interesting since for $a \neq 0$ the dependence on the extra coordinate can manifest itself as an apparent violation of the (4D) principle of equivalence.

The preceding two paragraphs dealt with two examples of the metric (4.60) as constrained by relations (4.61). We have looked at other examples, and note that in general the induced matter is an anisotropic fluid $\left(T_1^1 \neq T_2^2\right)$. We also note that while all solutions of $R_{AB} = 0$

and $R_{ABCD} = 0$ are mathematically related, their induced-matter contents are not the same. This because the 5D and 4D coordinate transformations

$$x^A \to \bar{x}^A(x^B) \qquad x^\alpha \to \bar{x}^\alpha(x^\beta) \tag{4.70}$$

are not equivalent. The first preserves $R_{AB} = 0$ and $R_{ABCD} = 0$, while the second preserves $G_{\alpha\beta} = 8\pi T_{\alpha\beta}$. But $T_{\alpha\beta}$ is an object derived from 5D geometry in induced-matter theory, and so changes under 5D coordinate transformations.

4.8 Conclusion

The induced matter (4.1) for any solution of the field equations depends on a choice of coordinates, so it is possible to start with flat 5D space (4.2) and use coordinate transformations to generate realistic cosmological models such as the standard one (4.12). The trick is to find appropriate coordinates to give matter of the kind indicated by observations, namely (4.13). However, some of that matter must be dark, and solitons with metric (4.14) are viable candidates which can be looked for and constrained in several ways. If the masses of particles (dark or otherwise) are related to the extra coordinate of Kaluza-Klein gravity, then the extra coordinate degree of freedom can be used either to specify a frame (4.23) where masses vary or a frame (4.27) where they are constants. In the former galaxies are static (comoving), while in the latter they move according to Hubble's law (non-comoving). Both frames are admissible, as 5D cosmology with induced matter is fully covariant.

Astrophysical systems of many types can be described by the spherically-symmetric metric (4.28), and a particular solution relevant to clusters of galaxies (4.31) can in principle be tested using gravitational lensing. More novel solutions can also be found which are of relevance. Thus (4.34) describes waves in a model with the equation of state of the classical

vacuum, and can by a change of parameter go over to a de Sitter inflationary model. And (4.45) describes a time-dependent soliton, whose matter can be used to describe inhomogeneities in the early universe that might eventually evolve into galaxies. However, we lack x^4-dependent 5D axially-symmetric solutions of the Kerr type, though we have cylindrically-symmetric solutions without singularities such as (4.50). The cellular structure revealed by surveys of galaxies, clusters and superclusters can in some approximation be described by solutions where the induced matter is concentrated on the surfaces of shells, as in (4.51), (4.53). That particular solution is 5D flat though 4D curved; and broad classes of solutions such as (4.60), (4.61) can be located which have this intriguing property.

In fact, the main thing we infer from the contents of this chapter is that the universe in the large, while it has matter and structure in 4D, may be empty and flat in 5D. This is certainly the conclusion we arrive at for the standard model. A corollary of this is that the big bang of physics in 4D is an artifact produced by an unfortunate choice of coordinates in 5D. Put in Eddingtonian terms, the big bang is not so much a property of the universe as a property of the way we describe the universe.

References (Chapter 4)

Abolghasem, G., Coley, A.A. McManus, D.J. 1996, J. Math. Phys. 37, 361.

Banerjee, A., Das, A., Panigrahi, D. 1995, Phys. Rev. D 51, 6816.

Billyard, A., Wesson, P.S. 1996a, Phys. Rev. D53, 731.

Billyard, A., Wesson, P.S. 1996b, Gen. Rel. Grav. 28, 129.

Bruckman, W. 1986, Phys. Rev. D34, 2990.

Bruckman, W. 1987, Phys. Rev. D36, 3674.

Chatterjee, S., Wesson, P.S., Billyard, A. 1997, Nuovo Cim., B112, 1667.

Chinea, F.J., Fernandez-Jambrina, L., Senovilla, J.M.M. 1992, Phys. Rev. D45, 481.

Clement, G. 1986, Gen. Rel. Grav. 18, 137.

Frolov, V.P., Zelnikov, A.I., Bleyer, U. 1987, Ann. Phys. (Leipzig) 44, 371.

Gott, J.R., Rees, M.J. 1987, Mon. Not. Roy. Astron. Soc. 227, 453.

Gross, D.J., Perry, M.J. 1983, Nucl. Phys. B226, 29.

Henriksen, R.N., Wesson, P.S. 1978, Astrophys. Space Sci 53, 429.

Horne, J.H., Horowitz, G.T. 1992, Phys. Rev. D46, 1340.

Kolb, E. 1989, Astrophys. J. 344, 543.

Lake, K., Musgrave, P.J., Pollney, D. 1995, GRTensor Version 1.19 (Dept. of Physics, Queen's University, Canada).

Liu, H., Wesson, P.S., Ponce de Leon, J. 1993, J. Math. Phys. 34, 4070.

Liu, H., Wesson, P.S. 1998, Gen. Rel. Grav., 30, 509.

Mashhoon, B., Liu, H., Wesson, P.S. 1994, Phys. Lett. B 331, 305.

Matos, T. 1994, J. Math. Phys. 35, 1302.

Myers, R.C., Perry, M.J. 1986, Ann. Phys. (New York) 172, 304.

Ponce de Leon, J. 1988, Gen. Rel. Grav. 20, 539.

Ponce de Leon, J. 1993, Gen. Rel. Grav. 25, 1123.

Ponce de Leon, J., Wesson, P.S. 1993, J. Math. Phys. 34, 4080.

Rindler, W. 1977, Essential Relativity (2nd ed., Springer, Berlin), 185.

Robertson, H.P., Noonan, T.W. 1968, Relativity and Cosmology (Saunders, Philadelphia), 367, 413.

Roque, W.L., Seiler, W.M. 1991, Gen. Rel. Grav. 23, 1151.

Ruiz, E., Senovilla, J.M.M. 1992, Phys. Rev. D45, 1995.

Senovilla, J.M.M. 1990, Phys. Rev. Lett. 64, 2219.

Tikekar, R., Patel, L.K., Dadhich, N. 1994, Gen. Rel. Grav. 26, 647.

Wesson, P.S. 1978, Cosmology and Geophysics (Oxford U. Press-Hilger, New York).

Wesson, P.S. 1991, Astrophys. J. 378, 466.

Wesson, P.S. 1992, Phys. Essays (Toronto) 5, 561.

Wesson, P.S., Liu, H., Lim, P. 1993, Phys. Lett. B 298, 69.

Wesson, P.S. 1994a, Astrophys. J. 436, 547.

Wesson, P.S. 1994b, Astrophys. J. 420, L49.

Wesson, P.S., Liu, H. 1995, Astrophys. J. 440, 1.

Wesson, P.S., Liu, H. 1998, Phys. Lett. B 432, 266.

Will, C.M. 1991, in Gravitation: A Banff Summer Institute (eds. Mann, R., Wesson, P.S., World Scientific, Singapore), 439.

5. 5D ELECTROMAGNETISM

"I felt a new kind of spark" (Madonna, Buenos Aires, 1995)

5.1 Introduction

Kaluza, we recall, realized that when there is no matter present the equations of 5D general relativity contain those of 4D Einstein gravity and 4D Maxwell electromagnetism (Kaluza 1921). Klein extended this approach to quantum theory (Klein 1926), and ever since Kaluza-Klein theory has been regarded as a unified theory of gravity and electromagnetism. More recently, though, various workers have shown that induced-matter theory as outlined in Chapter 2 explains the phenomenlogical properties of matter if the traditional geometry of Kaluza-Klein theory is broadened to include dependency on the extra coordinate. However, it is true that in this approach photons still occupy a special place. In the present chapter, we wish to see how electromagnetism fits into a general 5D geometry.

Thus we start with a 5D metric that depends on $x^4 = l$ and redo the algebra that puts it into a form where we recognize the electromagnetic potentials (Section 5.2). Then we work out the fully covariant form of the 5D geodesic equation (Section 5.3). We will find that the latter now contains x^4-dependent terms which are in principle important for testing 5D field theory. These terms are small in practice, however, so subsequently we drop x^4-dependence and present a class of exact solutions in 5D which extends the soliton class by including electric charge (Section 5.4). This class includes the case where the 4D part of the metric describes a Schwarzschild-like black hole, and to elucidate the effects of charge in 5D we then give an analysis of the electrodynamics involved in this case, finding that it leads to unexpected differences from 4D (Section 5.5). Finally, we return to the field equations and the question of

induced matter, and present the former in a way that allows us to evaluate the energy density of the electromagnetic field (Section 5.6). We will find that while 5D electromagnetism has been much studied, it yet contains some surprises.

5.2 Metrics and Potentials

The reduction of a 5D metric to a 4D one where the potentials can be recognized as those of gravity and electromagnetism has been done by various workers under the assumption of x^4-independence. In this and the following section, we follow Wesson and Ponce de Leon (1995) and look at the fully-covariant situation.

The 5D interval is given by

$$dS^2 = \gamma_{AB} dx^A dx^B \quad . \tag{5.1}$$

If we wish to investigate the 4D interval *ds* included in this, it is convenient to define a unit 5D vector Ψ^A, tangent to the fifth dimension:

$$\Psi^A \equiv \frac{\delta^A_4}{\sqrt{\varepsilon \gamma_{44}}} \quad . \tag{5.2}$$

This will enable us to split the 5D metric into a part parallel to Ψ^A and a 4D part orthogonal to it. Thus we define a projector:

$$g_{AB} \equiv \gamma_{AB} - \varepsilon \Psi_A \Psi_B \ , \tag{5.3}$$

where by (5.2)

$$g_{AB} = \gamma_{AB} - \frac{\gamma_{4A}\gamma_{4B}}{\gamma_{44}} \ . \tag{5.4}$$

This has

$$g_{44} = g_{4A} = 0 \ , \tag{5.5}$$

so the 5D line element (5.1) becomes the sum of a 4D line element and an extra part:

$$dS^2 = g_{\mu\nu}dx^\mu dx^\nu + \gamma_{44}\left(dx^4 + \frac{\gamma_{4\alpha}}{\gamma_{44}}dx^\alpha\right)^2 \tag{5.6}$$

$$ds^2 \equiv g_{\mu\nu}dx^\mu dx^\nu \ . \tag{5.7}$$

Here all of $g_{\mu\nu}$, γ_{44} and $\gamma_{4\alpha}$ can depend on x^4, and the signature is open via the choice $\varepsilon = \pm 1$ for this parameter. To see what is involved here, let us introduce a 4-vector and a scalar defined by

$$A_\mu \equiv \frac{\gamma_{\mu 4}}{\gamma_{44}} \tag{5.8}$$

$$\Phi^2 \equiv \varepsilon\gamma_{44} \ . \tag{5.9}$$

With these, (5.6) and (5.7) give

$$dS^2 = ds^2 + \varepsilon\Phi^2\left(dx^4 + A_\mu dx^\mu\right)^2 \ . \tag{5.10}$$

The 5D metric and the 4D metric are of course usually associated with groups of coordinate transformations which, as we noted in Section 4.7, are of the form

$$x^A \to \bar{x}^A(x^B) \tag{5.11}$$

$$x^\mu \to \bar{x}^\mu(x^\nu) \ . \tag{5.12}$$

We have seen before that physical quantities defined in 4D, while they are invariant under (5.12) are not invariant under (5.11), and the same holds here. However, in appropriate limits the quantities introduced above can be identified as usual. Thus $g_{\mu\nu}$ is associated with gravity, A_μ with electromagnetism, and Φ with a scalar interaction.

We will need some further relations in what follows. Thus (5.4) with $A = \alpha$ and $B = \beta$ when combined with (5.8) and (5.9) gives

$$\gamma_{\alpha\beta} = g_{\alpha\beta} + \varepsilon\Phi^2 A_\alpha A_\beta \ . \tag{5.13}$$

While if we take $\gamma^{AC}\gamma_{BC} = \delta^A_B$ with $A = \mu$, $B = 4$ and $C = \lambda, 4$ we obtain $\gamma^{\mu 4} = -\gamma^{\mu\lambda}\gamma_{4\lambda}/\gamma_{44}$. This can be used with (5.4) in the expansion of $\gamma^{AC}\gamma_{BC} = \delta^A_B$ with $A = \mu$, $B = \nu$ and $C = \lambda, 4$ to find

$$\gamma^{\mu\nu} = g^{\mu\nu} \ . \tag{5.14}$$

Also, since $\gamma^{\mu 4} = -\gamma^{\mu\lambda}A_\lambda$ we see that

$$\gamma^{4\mu} = -A^\mu . \tag{5.15}$$

And from (5.8) and (5.9) we have

$$\gamma_{4\mu} = \varepsilon\Phi^2 A_\mu \ . \tag{5.16}$$

These relations will be useful, especially (5.14) which will allow us to raise indices using either the 5D or the 4D metric.

5.3 Geodesic Motion

By minimizing the interval S we obtain the 5D geodesic equation

$$\frac{d^2x^A}{dS^2} + \Gamma^A_{BC}\frac{dx^B}{dS}\frac{dx^C}{dS} = 0 \ , \tag{5.17}$$

where the Christoffel symbol of the second kind is

$$\Gamma^A_{BC} = \frac{\gamma^{AD}}{2}\left(\frac{\partial\gamma_{BD}}{\partial x^C} + \frac{\partial\gamma_{CD}}{\partial x^B} + \frac{\partial\gamma_{BC}}{\partial x^D}\right) . \tag{5.18}$$

It is convenient to treat the 5 components of (5.17) in the order 4, 123, 0.

The $A = 4$ component of (5.17) can be rewritten as

$$\gamma_{4A}\frac{d^2x^A}{dS^2} + \Gamma_{4,CD}\frac{dx^C}{dS}\frac{dx^D}{dS} = 0 \ . \tag{5.19}$$

It may be verified by expansion, using (5.8) and (5.9), that (5.19) is equivalent to

$$\frac{dn}{dS}=\frac{1}{2}\frac{\partial\gamma_{CD}}{\partial x^4}\frac{dx^C}{dS}\frac{dx^D}{dS}\,, \tag{5.20}$$

where n is a scalar function:

$$n\equiv\varepsilon\Phi^2\left(\frac{dx^4}{dS}+A_\alpha\frac{dx^\alpha}{dS}\right). \tag{5.21}$$

From (5.20) we see that if the 5D metric γ_{CD} were independent of x^4, then n would be a constant of the motion. [This in the sense that $dn/dS=\left(\partial n/\partial x^A\right)\left(dx^A/dS\right)=0$ even though n could depend on x^A.] However, this will not in general be the case. We also note from (5.10) and (5.21) that

$$dS=\frac{ds}{\left(1-\varepsilon n^2/\Phi^2\right)^{1/2}}\,, \tag{5.22}$$

which gives the 5D interval in terms of its 4D part.

The A = 123 components of (5.17) give the equations of motion of a test particle in ordinary space. However, to obtain these explicitly we need to evaluate the following components of (5.18) using (5.13)-(5.16) above:

$$\Gamma^\mu_{44}=-\varepsilon\Phi\Phi^{,\mu}+\varepsilon\Phi\left(A^\mu\frac{\partial\Phi}{\partial x^4}+\Phi g^{\mu\lambda}\frac{\partial A_\lambda}{\partial x^4}\right) \tag{5.23}$$

$$\Gamma^\mu_{4\nu}=-\varepsilon\Phi\Phi^{,\mu}A_\nu-\frac{\varepsilon\Phi^2}{2}F^\mu{}_\nu+\frac{g^{\mu\lambda}}{2}\frac{\partial\gamma_{\lambda\nu}}{\partial x^4} \tag{5.24}$$

$$\hat{\Gamma}^\mu_{\alpha\beta}=\Gamma^\mu_{\alpha\beta}-\varepsilon\Phi\Phi^{,\mu}A_\alpha A_\beta-\frac{\varepsilon\Phi^2}{2}\left(A_\alpha F^\mu{}_\beta+A_\beta F^\mu{}_\alpha\right)+\frac{A^\mu}{2}\frac{\partial\gamma_{\alpha\beta}}{\partial x^4} \tag{5.25}$$

Here a semicolon as in $\Phi^{;\mu} = g^{\mu\nu}\Phi_{;\nu} = g^{\mu\nu}\left(\partial\Phi/\partial x^{\nu}\right)$ involves the usual 4D covariant derivative, $\Gamma^{\mu}_{\alpha\beta}$ is the usual 4D Christoffel symbol defined by a relation analogous to (5.18), and $F_{\mu\nu}$ is the usual 4D antisymmetric tensor

$$F_{\mu\nu} \equiv \frac{\partial A_{\nu}}{\partial x^{\mu}} - \frac{\partial A_{\mu}}{\partial x^{\nu}} = A_{\nu;\mu} - A_{\mu;\nu} \ . \tag{5.26}$$

To bring out both the gravitational and electromagnetic parts of the geodesic (5.17) in a recognizable form, we need to substitute for (5.23)-(5.26) and use the definition (5.21) for the quantity n. After considerable algebra, we find that (5.17) with $A = \mu$ can be written

$$\frac{d^2x^{\mu}}{dS^2} + \Gamma^{\mu}_{\alpha\beta}\frac{dx^{\alpha}}{dS}\frac{dx^{\beta}}{dS} = n F^{\mu}{}_{\alpha}\frac{dx^{\alpha}}{dS} + \varepsilon n^2\frac{\Phi^{;\mu}}{\Phi^3} - A^{\mu}\frac{dn}{dS} - g^{\mu\lambda}\frac{dx^4}{dS}\left(n\frac{\partial A_{\lambda}}{\partial x^4} + \frac{\partial g_{\lambda\nu}}{\partial x^4}\frac{dx^{\nu}}{dS}\right) . \tag{5.27}$$

If we replace total derivatives with respect to S by their counterparts with respect to s using (5.22), we obtain

$$\frac{d^2x^{\mu}}{ds^2} + \Gamma^{\mu}_{\alpha\beta}\frac{dx^{\alpha}}{ds}\frac{dx^{\beta}}{ds} = \frac{n}{\left(1 - \varepsilon n^2/\Phi^2\right)^{1/2}}\left[F^{\mu}{}_{\nu}\frac{dx^{\nu}}{ds} - \frac{A^{\mu}}{n}\frac{dn}{ds} - g^{\mu\lambda}\frac{\partial A_{\lambda}}{\partial x^4}\frac{dx^4}{ds}\right]$$

$$+ \frac{\varepsilon n^2}{\left(1 - \varepsilon n^2/\Phi^2\right)\Phi^3}\left[\Phi^{;\mu} + \left(\frac{\Phi}{n}\frac{dn}{ds} - \frac{d\Phi}{ds}\right)\frac{dx^{\mu}}{ds}\right] - g^{\mu\lambda}\frac{\partial g_{\lambda\nu}}{\partial x^4}\frac{dx^{\nu}}{ds}\frac{dx^4}{ds} \ . \tag{5.28}$$

This is the fully general equation of motion in Kaluza-Klein theory. We will look at the $\mu = 0$ component which is included in (5.28) below, but the $\mu = 123$ components deserve comment now. Thus, the l.h.s. of (5.28) if zero would define geodesic motion in 4D Einstein gravity, but the terms on the r.h.s. modify the motion. Of these, the 3 inside the first bracket are to do with electromagnetism. All 3 vanish if the electromagnetic potentials A_{μ} are zero, i.e. if $\gamma_{4\mu} = 0$ by (5.8). Of these 3 terms, the last 2 would vanish independently if the metric did not depend on x^4

(since then n would be a constant of the motion by the above so $dn/ds = 0$). This leaves the first term, which we recognize as the usual Lorentz force provided we write

$$\frac{q}{m} = \frac{n}{\left(1 - \varepsilon n^2 / \Phi^2\right)^{1/2}} \tag{5.29}$$

for the charge/mass ratio of the test particle.

The $A = 0$ component of (5.17) can usefully be treated separately from the preceding discussion (where it is included implicitly). This component can be written

$$\gamma_{0A} \frac{d^2 x^A}{dS^2} + \Gamma_{0,CD} \frac{dx^C}{dS} \frac{dx^D}{dS} = 0 \ . \tag{5.30}$$

It may be verified by expansion, using (5.13)-(5.16) and (5.21), that (5.30) is equivalent to

$$\frac{d}{dS}\left(g_{00} \frac{dx^0}{dS} + g_{0i} \frac{dx^i}{dS} \right) = -\frac{d\left(nA_0\right)}{dS} + \frac{1}{2} \frac{\partial \gamma_{CD}}{\partial x^0} \frac{dx^C}{dS} \frac{dx^D}{dS} \ . \tag{5.31}$$

Here $i = 123$. If as before we replace the total derivatives with respect to S by their counterparts with respect to s using (5.22), in all but the last factor (which we leave mixed for algebraic convenience), we obtain

$$\frac{d}{ds}\left\{ g_{00} \left(1 - \varepsilon n^2 / \Phi^2\right)^{1/2} \left[\frac{dx^0}{ds} + \frac{g_{0i}}{g_{00}} \frac{dx^i}{ds} \right] + nA_0 \right\} = \frac{1}{2} \frac{\partial \gamma_{CD}}{\partial x^0} \frac{dx^C}{ds} \frac{dx^D}{dS} \ . \tag{5.32}$$

This relation looks complicated. But it can be simplified greatly by using an argument analogous to what was done above in (5.2)-(5.7) and (5.19)-(5.21). Thus we define a unit 4D vector τ^μ, tangent to the time coordinate:

$$\tau^\mu \equiv \frac{\delta_0^\mu}{\sqrt{g_{00}}} \ . \tag{5.33}$$

We also define a projector onto the 3-space orthogonal to τ^μ as:

$$\lambda_{\mu\nu} \equiv \tau_\mu \tau_\nu - g_{\mu\nu} = \frac{g_{\mu 0} g_{\nu 0}}{g_{00}} - g_{\mu\nu} \ . \tag{5.34}$$

Then $\lambda_{00} = \lambda_{0j} = 0$ analogous to (5.5), and the 4D line element (5.7) becomes the sum of a time part and a space part:

$$ds^2 = g_{00}\left(dx^0 + \frac{g_{0i}}{g_{00}} dx^i \right)^2 - \lambda_{ij} dx^i dx^j \ . \tag{5.35}$$

This can be rewritten and factored to give

$$\frac{dx^0}{ds} = \frac{g_{0j}}{g_{00}} \frac{dx^j}{ds} = \frac{1}{\left(1 - \mathrm{v}^2\right)^{1/2} \left(g_{00}\right)^{1/2}} \ , \tag{5.36}$$

provided the square of the spatial 3-velocity is defined by $\mathrm{v}^2 \equiv \lambda_{ij} \mathrm{v}^i \mathrm{v}^j$ and the velocity itself is defined by

$$\mathrm{v}^i \equiv \frac{dx^i}{\left(g_{00}\right)^{1/2} \left[dx^0 + \left(g_{0i} / g_{00} \right) dx^i \right]} \ . \tag{5.37}$$

This is an element of ordinary space divided by an element of proper time in (5.35), and so is the appropriate definition of a 3-velocity analogous to the usual one for a 4-velocity. We can now use (5.36) to substitute for the term in square brackets in (5.32). The result is

$$\frac{de}{ds} = \frac{1}{2} \frac{\partial \gamma_{CD}}{\partial x^0} \frac{dx^C}{ds} \frac{dx^D}{dS} \ , \tag{5.38}$$

where e is a scalar function:

$$e \equiv \frac{\left(g_{00}\right)^{1/2}}{\left(1 - \mathrm{v}^2\right)^{1/2}} \left(1 - \frac{\varepsilon n^2}{\Phi^2} \right)^{1/2} + n A_0 \ . \tag{5.39}$$

The relations (5.38) and (5.39) are equivalent to the geodesic (5.30). From (5.38) we see that if the 5D metric γ_{CD} were independent of x^0, then e would be a constant of the motion. However, this will not in general be the case. We do note, though, that (5.39) bears a strong resemblance

to the expression for the energy $\in$ of a test particle in a static field in 4D general relativity (Landau and Lifshitz 1975):

$$\in = \frac{\left(g_{00}\right)^{1/2}}{\left(1-\mathrm{v}^2\right)^{1/2}} m + qA_0 \ . \tag{5.40}$$

A comparison together with previous relations shows that the mass of a test particle is related to the extra coordinate and the metric, while its electric charge is related to its rate of motion in the extra dimension.

We will make these identifications precise below, where we discuss the canonical metric (see Mashhoon, Liu and Wesson 1994 and Section 7.5). Here we comment that quantities like the mass of an object in astrophysics or the charge of a particle in laboratory physics are in practice defined through their equations of motion. What we have derived above is fully general, insofar as we have manipulated a metric (5.1) into a form (5.10) that separates gravity, electromagnetism and the scalar field but have not restricted in any way the dependency of the metric coefficients on the coordinates $x^A = (t, xyz, l)$. It is perhaps therefore not surprising that in 5D we can obtain greater insight into quantities like q and m than is possible in 4D. Also, our 5D geodesic components (5.20) for $A = 4$, (5.28) for $A = 123$ and (5.38) for $A = 0$ are typically 5D in nature. Thus there are extra terms present in the equation of motion for a particle in ordinary space (5.28) which are finite if there is x^4 dependency. Such terms can be used in principle to test the theory. For example, those terms in astrophysics would correspond to peculiar (non-Hubble) velocities for galaxies and clusters of galaxies. And the fact that these are observed to be less than or of the order of a few hundred km/sec (Lucey and Carter 1988; Goicoechea 1993; Calzetti and Giavalisco 1993) limits x^4 dependency. In the case where the metric does not depend on x^4, our relations go back to others in the literature (Leibowitz and Rosen 1973;

Kovacs 1984; Gegenberg and Kunstatter 1984; Davidson and Owen 1986; Ferrari 1989). We discuss elsewhere constraints that can be put on 5D theory by astrophysical dynamics, and here concentrate on electrodynamics.

5.4 Charged Solitons and Black Holes

The field equations of 5D Kaluza-Klein theory $R_{AB} = 0$ contain in an appropriate limit the 4D Einstein equations via $R_{\alpha\beta} = 0$ and the 4D Maxwell equations via $R_{4\alpha} = 0$ (see Section 1.5). There are many exact 5D solutions known which are static, (3D) spherically symmetric and describe a Schwarzschild-like gravitational field with or without an electromagnetic field (see Liu and Wesson 1996, 1997 for references). However, the algebraic equivalence or otherwise of these solutions to each other is unknown as their complexity precludes the identification of relevant coordinate transformations. Also, the physical interpretation of these solutions is difficult because the extra dimension is usually presumed to be compactified and the spacetime part is usually written in (3D) isotropic coordinates rather than the standard Schwarzschild coordinates used in the classical tests of relativity. Therefore, we will in this section give a class of solutions which is algebraically general yet physically easy to interpret.

Let us consider a metric and solution which are generalizations of those of the static solitons (Section 3.2):

$$dS^2 = \frac{(1-k)B^a}{(1-kB^{1-b})}dt^2 - B^{-a-b}\,dr^2 - r^2B^{1-a-b}d\Omega^2 - \frac{(B^b - kB^a)}{(1-k)}(dl + Adt)^2$$

$$A \equiv \frac{-\sqrt{k}(1-B^{a-b})}{(1-kB^{a-b})}$$

$$B \equiv 1 - \frac{2(1-k)M}{r}$$

$$1 = a^2 + b^2 + ab \ . \tag{5.41}$$

Here k is a dimensionless parameter whose nature we will elucidate below; A and B are dimensionless potentials that depend on k,r and the mass M at the centre of the 3-geometry; and a,b are dimensionless constants that obey the same consistency relation as before. The choice $a = 1$, $b = 0$, $k = 0$ makes the 4D part of the 5D metric Schwarzschild. The choice $a = 1$, $b = -1$, $k = 0$ makes the diagonal part of the 5D metric Chatterjee. The limit $k \rightarrow 0$ in general makes the off-diagonal part of the metric disappear and gives back the Gross-Perry/Davidson-Owen class of solitons. The last is a 2-parameter class, but (5.41) is a 3-parameter class, because it depends on k,M and one or the other of a,b. Clearly, we need to identify the new parameter k.

To do this, let us examine the case $a = 1$, $b = 0$, $k \neq 0$. Unlike the other solitons (which do not possess event horizons of the conventional sort), there is some justification for referring to this object as a black hole, because for $k \rightarrow 0$ the metric (5.41) becomes exactly that of Schwarzschild plus an extra flat piece. In general, the metric can now be written in the following form:

$$dS^2 = BE^{-1}dt^2 - B^{-1}dr^2 - r^2 d\Omega^2 - E(dl + Adt)^2$$

$$E \equiv \frac{(1-kB)}{(1-k)} = 1 + \frac{2Mk}{r}$$

$$A \equiv \frac{-\sqrt{k}(1-B)}{(1-kB)} = -\frac{2M\sqrt{k}}{Er}$$

$$B \equiv 1 - \frac{2M(1-k)}{r} = E - \frac{2M}{r} \ . \tag{5.42}$$

This is a 2-parameter *(k,M)* class, where A,B,E are the potentials. Of these, A is actually the electrostatic potential, as may be confirmed by comparing (5.42) with the Kaluza-Klein metric written in the standard form where the field equations yield Maxwell's equations. The electric

field has only a radial component (E_1, not to be confused with E above). In terms of the usual Faraday tensor $F_{\alpha\beta} \equiv \partial A_\beta / \partial x^\alpha - \partial A_\alpha / \partial x^\beta$ of (5.26), it is given by

$$E_1 = F_{10} = \frac{\partial A}{\partial r} = \frac{2M\sqrt{k}}{E^2 r^2} \ . \tag{5.43}$$

The associated invariant is

$$I \equiv F^{\alpha\beta} F_{\alpha\beta} = -2E(F_{10})^2 = -\frac{8M^2 k}{E^3 r^4} \ . \tag{5.44}$$

By (5.42) for $r \to \infty$, we have $B \to 1$, $E \to 1$ and $A \to -2M\sqrt{k}/r$. The last should be $-Q/r$ to agree with Coulomb's law at large distances from the source charge Q. This allows us to identify the parameter k with conventional units restored as

$$k = \frac{Q^2}{4GM^2} \ . \tag{5.45}$$

Using this in (5.42)-(5.44) allows us to sum up the quantities of interest (valid for any r) as follows:

$$E = 1 + \frac{Q^2}{2Mr}$$

$$A = -\frac{Q}{Er} \qquad B = E - \frac{2M}{r}$$

$$E_1 = \frac{Q}{E^2 r^2} \qquad I = -\frac{2Q^2}{E^3 r^4} \ . \tag{5.46}$$

Already here we can see that a quantity Q^2/Mr that depends on the <u>combination</u> of Q and M comes into the 5D solution, whereas only the quantities M/r and Q^2/r^2 that depend <u>separately</u> on Q and M come into the corresponding 4D (Reissner-Nordstrom) solution.

The preceding paragraph concerns the case $a = 1$, $b = 0$, $k \neq 0$. However, the choice $a = 1$, $b = 0$ is smoothly contained in the range $-2/\sqrt{3} < (a,b) < 2/\sqrt{3}$ allowed by the

consistency relation given by the last member of (5.41). Thus we identify the whole 3-parameter class of exact solutions (5.41) as charged solitons.

5.5 Charged Black Hole Dynamics

In this section, we will combine what we learned about geodesic motion in Section 5.3 with the solution presented in Section 5.4 to study the dynamics of a test particle with charge q and mass m in orbit around a 5D black hole with charge Q and mass M.

The metric for the latter is (5.42), which is independent of x^4 so n of (5.21) is a constant of the motion. That relation is then the first integral of the last component of the geodesic equation. The spacetime components are given by (5.28). The equations of motion are thus

$$\frac{dl}{dS}+A\frac{dt}{dS}=-\frac{n}{E} \tag{5.47}$$

$$\frac{d^2x^\alpha}{ds^2}+\Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds}=\frac{n}{\left(1+n^2/E\right)^{1/2}}F^\alpha{}_\beta\frac{dx^\beta}{ds}-\frac{n^2}{\left(1+n^2/E\right)}\frac{E_{,\beta}}{2E^2}\left(g^{\alpha\beta}-\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}\right). \tag{5.48}$$

By (5.29) or (5.48) directly, we obtain the asymptotic $\left(r\to\infty,\ E\to 1\right)$ value for the charge/mass ratio of the test particle as

$$\frac{q}{m}=\frac{n}{(1+n^2)^{1/2}} \qquad\qquad n=\frac{q}{\left(m^2-q^2\right)^{1/2}}\ . \tag{5.49}$$

The second term on the r.h.s. of (5.48) is an extra force not present in 4D electrodynamics, but it is tiny for classical as opposed to quantum situations because it depends on $(q/m)^2$, which with conventional units restored is $q^2/Gm^2 << 1$. For any r, we can rewrite (5.48) and (5.49) as

$$\frac{d^2x^\alpha}{ds^2}+\Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds}=\left(\frac{q}{mD}\right)F^\alpha{}_\beta\frac{dx^\beta}{ds}-\left(\frac{q}{mD}\right)^2\frac{E_{,\beta}}{2E^2}\left(g^{\alpha\beta}-\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}\right)$$

$$D\equiv\left(1-\frac{e^2Q^2}{2m^2MEr}\right)^{1/2}\ , \tag{5.50}$$

where $E=\left(1+Q^2/2Mr\right)$ as before. We can evaluate this for index $\alpha = 1$ in what might be called the standard approximation, specified by $|q/m|<<1$, $M/r<<1$, $u^{123}<<u^0\equiv\left(g^{00}\right)^{1/2}$. For this approximation, $\Gamma^1_{00}\cong M/r^2$, $F_0{}^1\cong Q/r^2$ and we obtain

$$m\frac{d^2r}{ds^2}=-\frac{mM}{r^2}+\frac{qQ}{r^2}\ , \tag{5.51}$$

which is of course true in many situations.

There are two cases of (5.47)-(5.49) which can be studied without approximation and are illustrative: (i) $Q = 0$, $q \neq 0$ and (ii) $Q \neq 0$, $q = 0$. For (i) we have by (5.45) $k = 0$ and we have by (5.42) $E = 1$, $A = 0$ and $B = 1\text{-}2M/r$. The metric (5.42) is

$$dS^2=\left(1-\frac{2M}{r}\right)dt^2-\frac{dr^2}{(1-2M/r)}-r^2d\Omega^2-dl^2\ , \tag{5.52}$$

which is just a neutral 4D Schwarzschild black hole plus an extra flat dimension. The motion of a charged particle around such a black hole from (5.47)-(5.49) is given by

$$\frac{dl}{dS}=-n\qquad\qquad n=\frac{q/m}{\left(1-q^2/m^2\right)^{1/2}} \tag{5.53}$$

$$\frac{d^2x^\alpha}{ds^2}+\Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds}=0\ . \tag{5.54}$$

In other words, for this case the constant of the motion associated with the extra dimension is essentially the charge-mass ratio of the test particle, and the 4D motion is geodesic in the Einstein sense. For (ii), we have by (5.49) $n = 0$, so by (5.46)-(5.48) there comes

$$\frac{dl}{dS}+A\frac{dt}{dS}=0 \qquad A=-\frac{Q}{r}\left(1+\frac{Q^2}{2Mr}\right)^{-1} \tag{5.55}$$

$$\frac{d^2x^\alpha}{ds^2}+\Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds}=0\ . \tag{5.56}$$

The 4D motion is again geodesic in the Einstein sense, while (5.55) implies that the metric (5.42) loses its extra part and becomes effectively 4D. Using (5.46), in the large-r approximation we find $B\equiv\left(1-2M/r+Q^2/2Mr\right)$ and $BE^{-1}\equiv\left(1-2M/r+Q^2/r^2\right)$, so the metric is

$$ds^2\equiv\left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)dt^2-\frac{dr^2}{\left(1-2M/r+Q^2/2Mr\right)}-r^2\,d\Omega^2\ . \tag{5.57}$$

This can be compared to the Reissner-Nordstrom solution of Einstein's theory:

$$ds^2=\left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)dt^2-\frac{dr^2}{\left(1-2M/r+Q^2/r^2\right)}-r^2\,d\Omega^2\,. \tag{5.58}$$

We see that they agree in g_{00} but not in g_{11}. That is, at large distances the 3-space associated with the '5D' black hole (5.57) is different from that associated with the 4D black hole (5.58). The difference will in practice be minor, since the mass equivalent of the energy associated with the central charge will be small in most situations $\left(GQ^2/c^4r^2<<1\right)$. However, this discrepancy prompts us to ask about the consistency of the electromagnetic field associated with the solutions (5.42). To be precise: The solutions (5.58) satisfy the 4D field equations $G_{\alpha\beta}=8\pi T_{\alpha\beta}$ in terms of the Einstein tensor and a standard energy-momentum tensor describing an electrostatic field, whereas (5.57) or (5.42) satisfy 5D field equations $R_{AB}=0$ with nothing on the right-hand side; so what is the effective energy-momentum tensor of the 5D solutions (5.42) if they are viewed in 4D?

The answer to this can be given here by anticipating results from the following section on the induced matter associated with non-diagonal metrics such as (5.10). For the charged black-

hole solutions (5.42) we find that the energy density associated with the electromagnetic field is given by

$$T_{00}^{em} = \frac{BM^2k}{E^3r^4} \to \frac{Q^2}{4r^4} \quad (r \to \infty) \ . \tag{5.59}$$

This has its conventional form, implying that the 5D solutions (5.42) do not show departures from the 4D ones in the far field even though they may in the near field.

The above concerns the electromagnetic field. Returning to dynamics, we expect small differences in the orbits of particles about the central mass, because the Lagrangian associated with the 5D charged black hole (5.42) differs from that associated with the 4D charged black hole (5.58) and also from that associated with the 5D neutral soliton (3.6). The latter was investigated extensively in Chapter 3 with regard to the classical tests of relativity in the solar system. And while we do not expect there to be significant electromagnetic effects in that situation, it is instructive to see how charges alter 5D dynamics. To examine this, let us rewrite (5.42) as

$$L = BE^{-1}\dot{t}^2 - B^{-1}\dot{r}^2 - r^2\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right) - E\left(\dot{l} + A\dot{t}\right)^2 \ . \tag{5.60}$$

Here $\dot{t} = dt/d\lambda$ etc., where λ is an affine parameter, and we can assume that the orbit is confined to the plane $\theta = \pi/2$ with $\dot{\theta} = 0$. Then from (5.60) we obtain in the usual way 3 constants of the motion:

$$\begin{aligned} j &\equiv BE^{-1}\dot{t} - E\left(\dot{l} + A\dot{t}\right)A \\ h &\equiv r^2\dot{\phi} \\ n &\equiv -E\left(\dot{l} + A\dot{t}\right) \end{aligned} \tag{5.61}$$

Rearranging these gives the 5-velocities in the case where $\lambda = S$ (so $L = 1$) as

$$\dot{t} = EB^{-1}(j - nA)$$

$$\dot{\phi} = hr^{-2}$$

$$\dot{l} = -nE^{-1} - EAB^{-1}(j - nA) \,. \tag{5.62}$$

For $Q = 0$ ($E = 1, A = 0$) these give $\dot{t} = B^{-1}j$, $\dot{\phi} = hr^{-2}$ and $\dot{l} = -n$, which are identical to the results of Chapter 3 for the case where the metric parameters are $a = 1$, $b = 0$. If also the test charge is $q = 0$, the 5D Lagrangian (5.60) becomes identical to the 4D one, so there is no difference in the dynamics. This applies obviously to photons, which perforce have $q = 0$; so the classical tests involving redshift, light bending and radar ranging are of minor interest. There remains, however, the question of what happens with massive particles which could conceivably have $q \neq 0$ and the analog of the perihelion test. To investigate this, we can take the Lagrangian (5.60) [with $\lambda = S$], substitute into it for the constants of motion (5.61) [with Q, $A \neq 0$ for now] and rearrange to get:

$$\frac{1}{B}\left(\frac{dr}{d\phi}\right)^2 = \frac{r^4}{h^2}\left[\frac{E}{B}(j - nA)^2 - \frac{n^2}{E} - 1\right] - r^2 \,. \tag{5.63}$$

Changing to a variable $u = 1/r$ as before, this yields

$$\left(\frac{du}{d\phi}\right)^2 = \frac{1}{h^2}\left[E(j - nA)^2 - \frac{n^2 B}{E} - B\right] - Bu^2 \,. \tag{5.64}$$

This formula could provide the basis for a detailed investigation of the orbit of a charged test particle about a charged source, but for now we look at the simpler case $Q = 0$ ($q \neq 0$), when (5.64) gives

$$\left(\frac{du}{d\phi}\right)^2 = \frac{(j^2 - n^2 - 1)}{h^2} + \frac{(n^2 + 1)2Mu}{h^2} + 2Mu^3 - u^2 \,. \tag{5.65}$$

Taking $d/d\phi$ of this and assuming the orbit is not perfectly circular $(du/d\phi \neq 0)$, we obtain

$$\frac{d^2u}{d\phi^2}=\frac{\left(1+n^2\right)}{h^2}M+3Mu^2-u\ . \tag{5.66}$$

This is a special case of equation (3.34) found before (it corresponds to $d = 1 + n^2$, $e = 0$ for the constants in the previous case). The test particle has a precession per orbit which in our notation is

$$\delta\phi=\frac{6\pi M^2}{h^2}\left[1+\frac{q^2}{\left(m^2-q^2\right)}\right]. \tag{5.67}$$

Here, the first term is the one quoted in the usual approach using Einstein's theory, while the second is an extra one that is necessarily positive. It exists because the Lagrangian (5.60) or metric (5.42) looks like the usual 4D one but augmented by a boost in the extra dimension [$\dot{l}=dl/dS=-n=-q(m^2-q^2)^{-1/2}$ by (5.62) and (5.49)]. The extra term in (5.67) could in principle be significant for a test particle with an appropriate charge/mass ratio.

The last relation (5.67) and others we have derived in this section have concentrated on the classical effects of charge, insofar as we have taken a class of 5D charged black-hole solutions (5.42) with a classical test particle. The latter has $q/G^{1/2}m<<1$ with conventional units restored. However, it would be feeble not to comment on the case $q/G^{1/2}m>>1$, which is typical of elementary particles and usually treated by quantum theory. That there is a problem in treating particles with large charge/mass ratios or fluids with large charge densities was appreciated by Kaluza (1921). The quantization of charge was attributed to compactification by Klein (1926), but this only partly addresses the problem with dynamics and runs into other difficulties (see Chapter 1). However, an interesting suggestion that may resolve the problem posed by particles with $q/G^{1/2}m>>1$ was made by Davidson and Owen (1986). It is simply that such particles move on spacelike as opposed to the usual timelike geodesics of a 5D space. An

inspection of our equations shows that it is possible to reformulate the dynamics in this case so that while $dS^2 < 0$ we still have $ds^2 > 0$, so 4D causality is maintained. This idea is intriguing but requires further study.

5.6 Field Equations and Induced Matter

The induced 4D energy-momentum tensor $T_{\alpha\beta}$ for metrics which satisfy the 5D field equations $R_{AB} = 0$ is usually evaluated algebraically (by rewriting the latter equations as $G_{\alpha\beta} = 8\pi T_{\alpha\beta}$) or by computer (by using programs that sort terms which are 4D and 5D in nature). However, both approaches have been developed primarily for metrics with $g_{4\alpha} = 0$ which describe matter with no electromagnetic potentials. Therefore, we will in this section carry out an analysis with these included that is analogous to that for neutral matter in Section 2.5. We aim particularly at evaluating the electromagnetic part of T_{00} for the charged 5D black holes discussed in Sections 5.4 and 5.5 above.

The line element $dS^2 = g_{AB}dx^A dx^B$ for electromagnetic problems is usually taken as (5.10), which it is convenient to rewrite in matrix form:

$$g_{AB} = \begin{bmatrix} \left(g_{\alpha\beta} - \Phi^2 A_\alpha A_\beta\right) & -\Phi^2 A_\alpha \\ -\Phi^2 A_\beta & -\Phi^2 \end{bmatrix} \qquad g^{AB} = \begin{bmatrix} g^{\alpha\beta} & -A^\alpha \\ -A^\beta & \left(-\Phi^{-2} + A^\mu A_\mu\right) \end{bmatrix}. \tag{5.68}$$

Here $A^\alpha = g^{\alpha\beta}A_\beta$ etc. as usual, and we use the same notation as before with hats added to the 4D parts of 5D quantities if there is a danger of confusion with the pure 4D quantities. For situations with electromagnetic fields, it is natural for reasons given above to consider the case where $g_{AB} = g_{AB}(x^\alpha)$. Then the nonzero 5D Christoffel symbols of the second kind are:

$$\hat{\Gamma}^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\beta\gamma} + \frac{1}{2}\Phi^2\left(A_\beta F^{\alpha}{}_{\gamma} + A_\gamma F^{\alpha}{}_{\beta}\right) + \Phi\Phi^{\alpha}A_\beta A_\gamma$$

$$\Gamma^{\alpha}_{44} = \Phi\Phi^{\alpha}$$

$$\Gamma^{\alpha}_{\beta 4} = \frac{1}{2}\Phi^2 F^{\alpha}{}_{\beta} + \Phi\Phi^{\alpha}A_\beta$$

$$\Gamma^{4}_{44} = -\Phi\Phi^{\lambda}A_\lambda$$

$$\Gamma^{4}_{4\beta} = -\frac{1}{2}\Phi^2 A^{\gamma}F_{\lambda\beta} - \Phi\Phi^{\lambda}A_\lambda A_\beta + \Phi^{-1}\Phi_\beta$$

$$\Gamma^{4}_{\alpha\beta} = \frac{1}{2}\left(A_{\alpha;\beta} + A_{\beta;\alpha}\right) - \frac{1}{2}\Phi^2 A^{\lambda}\left(A_\alpha F_{\lambda\beta} + A_\beta F_{\lambda\alpha}\right) - A_\alpha A_\beta \Phi\Phi^{\lambda}A_\lambda + \Phi^{-1}\left(A_\alpha\Phi_\beta + A_\beta\Phi_\alpha\right). \tag{5.69}$$

Here $\Phi_\alpha \equiv \Phi_{,\alpha}$ where a comma denotes the partial derivative, and $A_{\alpha;\beta} = A_{\alpha,\beta} - \Gamma^{\lambda}_{\alpha\beta}A_\lambda$ where a semicolon denotes the ordinary 4D covariant derivative. From (5.69) we have $\Gamma^{D}_{\alpha D} = \Gamma^{\lambda}_{\alpha\lambda} + \Phi^{-1}\Phi_\alpha$ and $\Gamma^{D}_{4D} = 0$, where a repeated index indicates summation. Using these and (5.69), we can work out the components of the 5D Ricci tensor:

$$\hat{R}_{\alpha\beta} = R_{\alpha\beta} + \frac{1}{2}\Phi^2 F_{\alpha\lambda}F_{\beta}{}^{\lambda} - \Phi^{-1}\Phi_{\alpha;\beta} + \frac{1}{2}A_\alpha\left(\Phi^2 F^{\gamma}{}_{\beta;\gamma} + 3\Phi\Phi^{\gamma}F_{\gamma\beta}\right)$$

$$+\frac{1}{2}A_\beta\left(\Phi^2 F^{\gamma}{}_{\alpha;\gamma} + 3\Phi\Phi^{\gamma}F_{\gamma\alpha}\right) + A_\alpha A_\beta\left(\frac{1}{4}\Phi^4 F^{\mu\nu}F_{\mu\nu} + \Phi\Phi^{\mu}{}_{;\mu}\right)$$

$$R_{4\alpha} = \frac{1}{2}\Phi^2 F^{\lambda}{}_{\alpha;\lambda} + \frac{3}{2}\Phi\Phi^{\lambda}F_{\lambda\alpha} + A_\alpha\left(\frac{1}{4}\Phi^4 F^{\mu\nu}F_{\mu\nu} + \Phi\Phi^{\mu}{}_{;\mu}\right)$$

$$R_{44} = \frac{1}{4}\Phi^4 F^{\mu\nu}F_{\mu\nu} + \Phi\Phi^{\alpha}{}_{;\alpha}\,. \tag{5.70}$$

These are written with the terms grouped in this fashion in order to simplify the field equations $R_{AB} = 0$, to which we will return below.

For now, we note that (5.70) can be used to obtain the 4D induced or effective energy-momentum tensor. First we form the 5D quantities

$$\hat{R} \equiv g^{AB} R_{AB} = R + \frac{1}{4}\Phi^2 F^{\mu\nu} F_{\mu\nu} - 2\Phi^{-1}\Phi^{\mu}{}_{;\mu} \quad . \tag{5.71}$$

$$\hat{R}^{\alpha\beta} \equiv \hat{g}^{\alpha A}\hat{g}^{\beta B} R_{AB} = R^{\alpha\beta} + \frac{1}{2}\Phi^2 F^{\alpha}{}_{\lambda} F^{\beta\lambda} - \Phi^{-1}\Phi^{\alpha;\beta} \quad . \tag{5.72}$$

Then we find that the 4D part of the 5D Einstein tensor and the usual 4D Einstein tensor are related by:

$$\hat{G}^{\alpha\beta} \equiv \left[\hat{R}^{\alpha\beta} - \frac{1}{2}\hat{g}^{\alpha\beta}\hat{R}\right]$$

$$= \left[R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R\right] + \frac{1}{2}\Phi^2\left(F^{\alpha}{}_{\lambda}F^{\beta\lambda} - \frac{1}{4}g^{\alpha\beta}F^{\mu\nu}F_{\mu\nu}\right) - \Phi^{-1}\left(\Phi^{\alpha;\beta} - g^{\alpha\beta}\Phi^{\mu}{}_{;\mu}\right). \tag{5.73}$$

If we now impose the field equations $R_{AB} = 0$ or $G_{AB} = 0$, the first 10 may be rewritten using (5.73) in the form of Einstein's equations:

$$R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R = 8\pi\left(T_{em}^{\alpha\beta} + T_{s}^{\alpha\beta}\right)$$

$$T_{em}^{\alpha\beta} \equiv -\frac{1}{2}\Phi^2\left(F^{\alpha}{}_{\lambda}F^{\beta\lambda} - \frac{1}{4}g^{\alpha\beta}F^{\mu\nu}F_{\mu\nu}\right)$$

$$T_{s}^{\alpha\beta} \equiv \Phi^{-1}\left(\Phi^{\alpha;\beta} - g^{\alpha\beta}\Phi^{\mu}{}_{;\mu}\right) \quad . \tag{5.74}$$

The last two quantities are the energy-momentum tensors as determined mainly by the electromagnetic and scalar fields, respectively.

Returning to the full set of 15 field equations $R_{AB} = 0$, these may by (5.70) be written as 4 neat sets of 10, 4, 1:

$$R_{\alpha\beta} = -\frac{1}{2}\Phi^2 F_{\alpha\lambda}F_{\beta}{}^{\lambda} + \Phi^{-1}\Phi_{\alpha;\beta}$$

$$F^{\lambda}{}_{\alpha;\lambda} = -3\Phi^{-1}\Phi^{\lambda}F_{\lambda\alpha}$$

$$\Phi^{\alpha}{}_{;\alpha} = -\frac{1}{4}\Phi^3 F^{\mu\nu}F_{\mu\nu} \quad . \tag{5.75}$$

In general, these may be regarded as Einstein-like equations for spacetime curved by electromagnetic and scalar fields, Maxwell-like equations where the source depends on the scalar field, and a wave-like equation for the scalar potential. They have wide applicability.

For present purposes, we can look at (5.75) for the case where the 4D part of the metric and the electromagnetic potential have the forms

$$ds^2 = e^{\nu}dt^2 - e^{\lambda}dr^2 - e^{\mu}r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

$$A_\alpha = \left(A_0,0,0,0\right) \ . \tag{5.76}$$

Here the coefficients ν, λ, μ and A_0 depend only on r. In this case, the last member of (5.75) reduces to

$$\Phi'' + \left(\frac{\nu' - \lambda' + 2\mu'}{2} + \frac{2}{r}\right)\Phi' - \frac{1}{2}\Phi^3 e^{-\nu}\left(A_0'\right)^2 = 0 \ , \tag{5.77}$$

where a prime denotes the derivative with respect to r. Also the middle member of (5.75) can be once integrated to give

$$A_0' = -\frac{C}{r^2\Phi^3}\, e^{(\nu+\lambda-2\mu)/2} \ , \tag{5.78}$$

where C is an arbitrary constant. The first member of (5.75) gives 3 relations:

$$\nu'' + \nu'\left(\frac{\nu' - \lambda' + 2\mu'}{2} + \frac{2}{r}\right) = -\frac{\Phi'}{\Phi}\nu' - \Phi^2 e^{-\nu}\left(A_0'\right)^2$$

$$\nu'' + 2\mu'' + \frac{(\nu')^2 + 2(\mu')^2 - \lambda'(\nu' + 2\mu')}{2} + \frac{4\mu' - 2\lambda'}{r} = -\Phi^2 e^{-\nu}\left(A_0'\right)^2 - 2\Phi^{-1}\left(\Phi'' - \frac{\lambda'}{2}\Phi'\right)$$

$$1 - \frac{1}{2}e^{\mu-\lambda}r^2\left[\mu'' + \frac{\mu'(\nu' - \lambda' + 2\mu')}{2} + \frac{\nu' - \lambda' + 4\mu'}{r} + \frac{2}{r^2}\right] = \frac{1}{2}e^{\mu-\lambda}r^2\left(\mu' + \frac{2}{r}\right)\frac{\Phi'}{\Phi} \ . \tag{5.79}$$

The 5 equations (5.77)-(5.79) constitute a complete set for the 5 unknowns Φ, A_0, ν, λ and μ. It may be verified by hand that the charged soliton solutions (5.41) satisfy (5.77)-(5.79), or by computer that they satisfy the equivalent set $R_{AB} = 0$.

The charged black-hole solutions (5.42) are an algebraically simple subset of (5.41), and for reasons given in Section 5.5 we are particularly interested in evaluating the energy density of the electrostatic field in this case. Putting (5.42) into (5.74) and using (5.43), (5.44) gives

$$T_{00}^{em} = -\frac{1}{2}\Phi^2\left[F_{0\lambda}F_0{}^{\lambda} - \frac{1}{4}g_{00}F^{\mu\nu}F_{\mu\nu}\right]$$

$$= -\frac{1}{2}E\left[g^{11}(F_{01})^2 - \frac{1}{4}g_{00}F^{\mu\nu}F_{\mu\nu}\right]$$

$$= \frac{BM^2k}{E^3r^4} \quad . \tag{5.80}$$

By (5.45) $k = Q^2/4M^2$ and by (5.46) for $r \to \infty$ we have $E \to 1$, $B \to 1$, so $T_{00}^{em} \to Q^2/4r^4$. This result, quoted as (5.59) in Section 5.5, is of conventional form and shows that the induced-matter approach is valid for electromagnetic fields.

5.7 Conclusion

The 5D metric with gravitational, electromagnetic and scalar potentials is commonly taken as (5.10) to distinguish these fields. The geodesic equation gives a new scalar associated with motion in the extra dimension (5.21), which comparison with the motion in spacetime (5.28) reveals to be connected with the charge/mass ratio of a test particle (5.29). However, the new scalar is not a constant of the motion unless the metric is independent of x^4, and the charge/mass ratio is in general a function of all 5 coordinates. The time component of the 5D geodesic defines a scalar (5.39) analogous to the particle energy (5.40) in general relativity. The

static, neutral solitons which we met before as the standard 1-body solutions in 5D can be generalized to a 3-parameter class with charge (5.41). In the case where the embedded 4-space in the neutral limit is Schwarzschild, we can speak in the general case of a 2-parameter class of charged 5D black holes (5.42). The motion of a test particle in orbit around such an object shows interesting effects, but even when the 5D metric becomes effectively 4D (5.57) it does not exactly reproduce the Riessner-Nordstrom metric (5.58) of general relativity. The field equations for x^4-independent electromagnetic problems are (5.75) and have wide applicability; but the energy density of the electrostatic field for the charged 5D black holes as calculated by the induced matter approach (5.80) is somewhat conventional.

Notwithstanding this, there are some odd results connected with 5D electromagnetism. The fact that these are small in the classical domain suggests that future work in this area should focus on the particle domain.

References (Chapter 5)

Calzetti, D., Giavalisco, M. 1993, Astrophys. J. 406, 388.

Davidson, A., Owen, D.A. 1986, Phys. Lett. B177, 77.

Ferrari, J.A. 1989, Gen. Rel. Grav. 21, 683.

Gegenberg, J., Kunstatter, G. 1984, Phys. Lett. A106, 410.

Goicoechea, L.J. 1993, Astron. Astrophys. 269, L9.

Kaluza, T. 1921, Sitz. Preuss. Akad. Wiss. 33, 966.

Klein, O. 1926, Z. Phys. 37, 895.

Kovacs, D. 1984, Gen. Rel. Grav. 16, 645.

Landau, L.D., Lifshitz, E.M. 1975, The Classical Theory of Fields (Pergamon, Oxford), 251.

Leibowitz, E., Rosen, N. 1973, Gen. Rel. Grav. 4, 449.

Liu, H., Wesson, P.S. 1996, Phys. Lett. B381, 420.

Liu, H., Wesson, P.S. 1997, Class. Quant. Grav. 14, 1651.

Lucey, D., Carter, D. 1988, Mon. Not. Roy. Astron. Soc. 235, 1177.

Mashhoon, B., Liu, H., Wesson, P.S. 1994, Phys. Lett. B331, 305.

Wesson, P.S., Ponce de Leon, J. 1995, Astron. Astrophys. 294, 1.

6. THE CANONICAL METRIC AND THE FIFTH FORCE

"To be believable it has to be measurable" (Francis Everitt, Stanford, 1996)

6.1 Introduction

In Chapter 2 we analysed several solutions of the 5D field equations $R_{AB} = 0$, some of which depended on the extra coordinate $x^4 = l$, and ended by giving a general prescription for x^4-dependent metrics that allows us to evaluate their effective or induced 4D matter. In Chapter 6 we started with an x^4-dependent form of the metric suited to 5D electromagnetism, but then concentrated on the x^4-independent case in order to obtain results on 4D electrodynamics. Both of these approaches made use of the algebraic freedom inherent in unrestricted 5D Riemannian geometry to choose the 5 coordinates x^A freely, with resultant simplifications for the metric tensor g_{AB} and other objects such as the Ricci tensor R_{AB} that figure in the theory. In general relativity, this is commonly called a choice of coordinates, but in particle physics the analogous procedure is called a choice of gauge. In the present chapter, we will start with a general discussion of gauges, but then concentrate on a new choice which is especially suited to dynamics. The main result is that there is an extra or fifth force inherent in 5D theory which has no counterpart in 4D general relativity.

6.2 Gauges in Kaluza-Klein Theory

The choice of a gauge in physics, like the choice of a map projection in geography, depends on what one wants to achieve. And the vast majority of gauges, if chosen arbitrarily, are therefore of little practical use. However, the original electromagnetic gauge of Kaluza-Klein theory, which consists in $g_{\alpha\beta} = g_{\alpha\beta}(x^\gamma)$, $g_{4\alpha} \equiv A_\alpha(x^\gamma)$, $g_{44} = -1$, is useful, because while it

imposes the x^4-independent or "cylinder" condition it allows one to identify the potentials of Maxwell's theory. Likewise the "matter" gauge wherein $g_{\alpha\beta} = g_{\alpha\beta}(x^A)$, $g_{4\alpha} = 0$, $g_{44} = g_{44}(x^A)$ is useful, because it allows one to go from $R_{AB} = 0$ to $G_{\alpha\beta} = 8\pi T_{\alpha\beta}$ and identify the 4D Einstein tensor and the induced energy-momentum tensor with its associated properties of matter.

A natural question, and one we wish to answer before proceeding, concerns the status of the 5D analog of the 4D harmonic gauge. The latter is extensively used in general relativity, notably to describe gravitational waves in the weak-field limit of Einstein's theory (see e.g. Weinberg 1972). The corresponding 5D harmonic gauge, in terms of the Christoffel symbol and the determinant of the metric, is given by

$$g^{AB}\Gamma^{C}_{AB} = -\frac{1}{g^{1/2}}\frac{\partial}{\partial x^{D}}\left(g^{1/2}\, g^{CD}\right) = 0 \ . \tag{6.1}$$

These are 5 conditions, but since Γ^{C}_{AB} is not a tensor the conditions (6.1) are not 5D covariant. This is the classical analog of the situation in quantum field theory involving non-covariant gauges (Leibbrandt 1994). In physical terms, fields which are split on one hypersurface of the metric will in general be mixed on another hypersurface, implying a problem when comparing 5D and 4D physics. By analogy with the latter, we assume that the metric is close to flat:

$$g_{AB} = \eta_{AB} + h_{AB}, \qquad \eta_{AB} = \pm 1, \qquad |h_{AB}| << 1 \ . \tag{6.2}$$

The gauge conditions (6.1) with (6.2) now read

$$\frac{\partial}{\partial x^{A}}\left(h^{A}_{\ B}\right) = \frac{1}{2}\frac{\partial}{\partial x^{B}}\left(h^{A}_{\ A}\right) \ . \tag{6.3}$$

We note in passing that these conditions are very restrictive, being first-order relations that must be applied to second-order field equations. The latter in the approximation (6.2) become

$$\Box h_{AB} = 0, \qquad \Box \equiv \eta^{AB} \frac{\partial^2}{\partial x^A \partial x^B} \quad . \tag{6.4}$$

This 5D box operator can be written in terms of its 4D counterpart and extra terms, so the field equations (6.4) read

$$\Box_4 h_{AB} = -2\eta^{\alpha 4} \frac{\partial^2 h_{AB}}{\partial x^\alpha \partial x^4} - \eta^{44} \frac{\partial^2 h_{AB}}{\partial x^4 \partial x^4} \quad . \tag{6.5}$$

That is, free waves in 5D become waves with sources in 4D. Regarding the r.h.s. of (6.5) as a 4D source, solutions can be written at least in principle using Green's functions (Myers and Perry 1986; it may be necessary to carry out an imaginary coordinate transformation or Wick rotation depending on the sign of the last part of the metric). But while the 5D harmonic gauge shares the general property of modern Kaluza-Klein theory, namely that source-free equations in 5D become ones with sources in 4D, it is clear that it has problems with application to physics.

It is possible, if physics is not the prime concern, to decompose the 15 5D field equations $R_{AB} = 0$ into sets of 10, 4, 1 4D field equations without the imposition of a specific gauge. This involves using an extended version of the lapse and shift technique commonly employed in general relativity (Misner, Thorne and Wheeler 1973; Hawking and Ellis 1973; Wald 1984). A preliminary application of this confirms what may be inferred from previous work, namely that the 10 Einstein-like equations describe physics on a 4D hypersurface, while the 4 Maxwell-like or conservation equations and the 1 wave-like equation for the scalar field depend on how that surface is embedded in 5D (Sajko, Wesson and Liu 1998; see also Section 2.5). However, it is still true that application to a particular problem in physics requires the choice of a specific gauge or form of the metric.

A form which we have met before in several contexts can succinctly be restated here as:

$$dS^2 = l^2 ds^2 - \Phi^2 dl^2$$

$$ds^2 \equiv g_{\alpha\beta}(x^\alpha, l) dx^\alpha dx^\beta \quad , \qquad \Phi = \Phi(x^\alpha, l). \tag{6.6}$$

That is, the 5D line interval is rewritten as the sum of a 4D part relevant to gravity and an extra part involving the scalar field, but there are no potentials associated with electromagnetism. However, the latter restriction uses up only 4 of the available 5 coordinate degrees of freedom, so (6.6) still has a degree of generality and can accordingly be applied in a number of contexts. It contains, for example, the Ponce de Leon metrics of standard 5D cosmology discussed in Chapters 2 and 4. Following the methods outlined there, the field equations $R_{AB} = 0$ for (6.6) may be expressed as follows:

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta}$$

$$8\pi T_{\alpha\beta} \equiv \frac{\Phi_{\alpha;\beta}}{\Phi} + \frac{1}{2\Phi^2}\left\{\frac{\dot{\Phi}}{\Phi}\left(2l\, g_{\alpha\beta} + l^2 \dot{g}_{\alpha\beta}\right) - l^2 \ddot{g}_{\alpha\beta} \quad + l^2 g^{\lambda\mu} \dot{g}_{\alpha\lambda} \dot{g}_{\beta\mu} - 4l \dot{g}_{\alpha\beta} - \frac{l^2}{2} g^{\mu\nu} \dot{g}_{\mu\nu} \dot{g}_{\alpha\beta}\right.$$

$$\left. + \left[6 + 2l g^{\mu\nu} \dot{g}_{\mu\nu} + \frac{l^2}{4} \dot{g}^{\mu\nu} \dot{g}_{\mu\nu} + \frac{l^2}{4}\left(g^{\mu\nu} \dot{g}_{\mu\nu}\right)^2\right] g_{\alpha\beta}\right\}$$

$$P^{\beta}_{\alpha;\beta} = 0$$

$$P_{\alpha\beta} \equiv \frac{1}{2\Phi}\left\{l^2 \dot{g}_{\alpha\beta} - \left[6l + l^2\left(g^{\mu\nu} \dot{g}_{\mu\nu}\right)\right] g_{\alpha\beta}\right\} .$$

$$\Box\Phi = \frac{l^2}{\Phi}\left\{\frac{1}{l} g^{\mu\nu} \dot{g}_{\mu\nu} + \frac{1}{4} \dot{g}^{\mu\nu} \dot{g}_{\mu\nu} + \frac{1}{2} g^{\mu\nu} \ddot{g}_{\mu\nu} - \frac{\dot{\Phi}}{\Phi}\left[\frac{4}{l} + \frac{1}{2} g^{\mu\nu} \dot{g}_{\mu\nu}\right]\right\}$$

$$\Box\Phi \equiv g^{\mu\nu} \Phi_{\mu;\nu} \quad . \tag{6.7}$$

Here as elsewhere, $\Phi_\mu \equiv \partial\Phi / \partial x^\mu$, a semicolon denotes the ordinary 4D covariant derivative, and a star denotes the partial derivative w.r.t. $x^4 = l$. These equations have wide applicability, but the choice of coordinates (6.6) from which they derive has not traditionally been given a name.

In view of more recent developments, it is appropriate to refer to (6.6) as the "semi-canonical" gauge.

The true canonical gauge is arrived at by using the remaining degree of coordinate freedom to set $\Phi^2(x^A)$ in (6.6) to unity (Mashhoon, Liu and Wesson 1994). Introducing a constant length L to preserve physical dimensions, we then have

$$dS^2 = \frac{l^2}{L^2} g_{\alpha\beta}(x^\alpha, l)\, dx^\alpha dx^\beta - dl^2 \ . \tag{6.8}$$

This metric leads to significant simplification of the induced-matter relations and also of the geodesic equations (5.28). The price one pays for this is that the 4D sector now harbors effects of the fifth dimension, which in general show up as departures from the usual laws of motion. However, such effects are Machian in the sense that they can be viewed as the influence of the rest of the universe on the properties of a local particle (Jordan 1955; Bekenstein 1977; Wesson 1978; Shapiro 1993). Such cosmological effects can be couched in terms of the time-variability of particle rest mass or the gravitational "constant", the choice being largely irrelevant because the two parameters occur in conjunction in gravitational problems. Limits on time-variability of the latter may be used to constrain the former, so we know that dynamical effects of the canonical gauge are at most of order 10^{-11} yr^{-1} (Will 1992). However, if the world is 5D in nature and depends significantly on the extra dimension, these effects are necessarily finite, and can be expressed in terms of a fifth force which is quite different from other known forces (Mashhoon, Wesson and Liu 1998). In view of the fundamental importance of this issue, we proceed to examine the main aspects of it in the following sections.

6.3 The Field Equations and the Cosmological Constant

Let us apply the canonical gauge (6.8) to the field equations (6.7). Then the middle member remains virtually unchanged, the last member becomes a constraint, and the first set reads

$$G_{\alpha\beta}=\frac{1}{2L^2}\left[-l^2\overset{\bullet\bullet}{g}_{\alpha\beta}+l^2 g^{\lambda\mu}\overset{\bullet}{g}_{\alpha\lambda}\overset{\bullet}{g}_{\beta\mu}-4l\overset{\bullet}{g}_{\alpha\beta}-\frac{l^2}{2}g^{\mu\nu}\overset{\bullet}{g}_{\mu\nu}\overset{\bullet}{g}_{\alpha\beta}\right]$$

$$+\frac{1}{2L^2}\left[6+2l\,g^{\mu\nu}\overset{\bullet}{g}_{\mu\nu}+\frac{l^2}{4}g^{\mu\nu}\overset{\bullet}{g}_{\mu\nu}+\frac{l^2}{4}\left(g^{\mu\nu}\overset{\bullet}{g}_{\mu\nu}\right)^2\right]g_{\alpha\beta}\,. \tag{6.9}$$

When $\overset{\bullet}{g}_{\alpha\beta}=0$, as in general relativity, these equations read just $G_{\alpha\beta}=3g_{\alpha\beta}/L^2$. These are Einstein's equations with a cosmological constant $\Lambda=3/L^2$, which is small if L is a large cosmological length. We see that from the viewpoint of Kaluza-Klein theory, the cosmological constant of Einstein's theory is a kind of artifact produced by a certain reduction from 5D to 4D.

The preceding paragraph identified the cosmological constant from the field equations in a procedure which appears to be straightforward (Mashhoon, Wesson and Liu 1997). However, we recall that in general relativity this parameter can be regarded as defining a density and pressure for the vacuum via $\rho=\Lambda/8\pi=-p$ with the characteristic equation of state $p+\rho=0$ (see Section 1.3). These properties of matter can be regarded as part of the energy-momentum tensor $T_{\alpha\beta}$, and it is really the equation of state which defines the vacuum. In modern Kaluza-Klein theory, the field equations are $R_{AB}=0$. These are scale-free, and it would run counter to the philosophy of the theory to introduce a constant like Λ with the physical dimensions of L^{-2} or T^{-2}. But in the induced-matter picture, the effective energy-momentum tensor will in general contain terms which we identify as ordinary matter and other terms which we by default regard as relating to the vacuum. The question then arises of whether it is possible to interpret exact 5D

solutions whose equation of state is $p+\rho=0$ in terms of something like the cosmological constant.

To examine this, let us consider 2 exact solutions which have appeared in the literature and which we will return to later in a different form (Mashhoon, Liu and Wesson 1994; Liu and Mashhoon 1995; Sections 7.2, 7.3). We give them here in a scale-free or self-similar form (Henriksen, Emslie and Wesson 1983):

$$dS^2=\frac{l^2}{3}\left[\left(1-\frac{2\alpha}{R}-\frac{R^2}{3}\right)dT^2-\left(1-\frac{2\alpha}{R}-\frac{R^2}{3}\right)^{-1}dR^2-R^2d\Omega^2\right]-dl^2 \tag{6.10}$$

$$dS^2=\frac{l^2}{3}\left[dT^2-e^{2T/\sqrt{3}}\left(dR^2+R^2d\Omega^2\right)\right]-dl^2\ . \tag{6.11}$$

Both metrics are canonical, and in both the coordinates *T, R* are dimensionless, with the physical dimension of *dS* carried by the coordinate l. For (6.10), the Kretschmann scalar is $K\equiv R^{ABCD}R_{ABCD}=432\alpha^2/l^4R^6$. This and all components of the Riemann-Christoffel tensor are zero if the dimensionless constant α is zero. For (6.11), all components of the Riemann-Christoffel tensor are zero anyway. Thus both (6.10) and (6.11) can describe 4D curved spaces embedded in flat 5D spaces (see Section 4.7). When we look at the induced energy-momentum tensor, we find that for both solutions $8\pi T_{\alpha\beta}=g_{\alpha\beta}$ where $g_{\alpha\beta}$ refers to the 4D part of the 5-metric. This we recognize as the signature of a vacuum; but an ambiguity exists as to whether we should raise an index using the 4-metric or the 5-metric. The former procedure does not make much physical sense, while the latter procedure gives $8\pi\rho\equiv 8\pi T_0^0=3/l^2$, $8\pi p\equiv -8\pi T_1^1=-3/l^2$ (Wesson 1997). Thus the equation of state is $p+\rho=0$, but both properties of matter depend on the extra coordinate, leading to fundamental questions about the nature of the cosmological constant.

This parameter is, of course, at the centre of a muddle in 4D physics. The cosmological-constant problem basically consists in a contradiction between the small value of this parameter as derived from general relativity and cosmological data, and the large value inferred from quantum field theory and its zero-point fields (see Weinberg 1989 and Ng 1992 for reviews). Based on what was stated above, a resolution of this problem is available at least in principle in 5D physics. It is simply that the induced energy-momentum tensor follows from a solution of local field equations, so the density and pressure of the vacuum depend on the situation and do not necessarily define a universal constant.

6.4 The Equations of Motion and the Fifth Force

In this section we will derive the equations of motion using a Lagrangian approach for the canonical metric (6.8). This involves a choice of coordinates about which we wish to make a few preliminary comments. The coordinates in (6.8) can be constructed by taking a 4D hypersurface in the 5D manifold, and regarding the lines normal to this hypersurface as the extra coordinate. These lines will be geodesics, and proper length along them will be the extra coordinate $x^4 = l$. This method of constructing a coordinate system is the 5D analog of how the synchronous coordinate system of general relativity is set up in 4D. It is based on the assumption that one can construct a 4D hypersurface in 5D with no obstruction. This coordinate system breaks down only if the coordinate lines in the fifth dimension cross. Therefore, apart from pathological situations, our coordinates are always admissible within a finite interval along the fifth dimension.

The equations of motion are obtained by minimizing the distance between two points in 5D, via $\delta\left[\int dS\right]=0$. If λ is an arbitrary affine parameter along the path, this relation can be

written $\delta\left[\int L d\lambda\right]=0$, where using (6.8) the quantity which is commonly referred to as the Lagrangian is

$$L \equiv \frac{dS}{d\lambda}=\left[\frac{l^2}{L^2} g_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda}-\left(\frac{dl}{d\lambda}\right)^2\right]^{1/2} . \tag{6.12}$$

The path in 5D is described by $x^\alpha = x^\alpha(\lambda), l = l(\lambda)$. The momenta and the equations of motion are then given as usual by

$$p_\alpha = \frac{\partial L}{\partial(dx^\alpha / d\lambda)} \qquad p_l = \frac{\partial L}{\partial(dl / d\lambda)} \tag{6.13}$$

$$\frac{dp_\alpha}{d\lambda}=\frac{\partial L}{\partial x^\alpha} \qquad \frac{dp_l}{d\lambda}=\frac{\partial L}{\partial l} . \tag{6.14}$$

Defining $u^\alpha \equiv dx^\alpha / d\lambda$, $u^l \equiv dl / d\lambda$ and $\theta \equiv L^2 = \left(l^2 / L^2\right) g_{\alpha\beta} u^\alpha u^\beta - (u^l)^2$, we then have for the momenta

$$p_\alpha = \left(\frac{l^2}{L^2}\right) \frac{g_{\alpha\beta} u^\beta}{\theta^{1/2}} \qquad p_l = -\frac{u^l}{\theta^{1/2}} . \tag{6.15}$$

The corresponding equations of motion are

$$\frac{d}{d\lambda}\left(g_{\alpha\beta} u^\beta\right)+\frac{2}{l}\left(\frac{dl}{d\lambda}\right) g_{\alpha\beta} u^\beta - g_{\alpha\beta} u^\beta \left(\frac{1}{2\theta}\right)\frac{d\theta}{d\lambda} = \frac{1}{2}\frac{\partial g_{\beta\gamma}}{\partial x^\alpha} u^\beta u^\gamma \tag{6.16}$$

$$\frac{d^2 l}{d\lambda^2}-\left(\frac{1}{2\theta}\right)\frac{d\theta}{d\lambda}\frac{dl}{d\lambda}=-\frac{l}{L^2} g_{\alpha\beta} u^\alpha u^\beta - \frac{1}{2}\left(\frac{l^2}{L^2}\right)\frac{\partial g_{\beta\gamma}}{\partial l} u^\beta u^\gamma . \tag{6.17}$$

We interject at this point the observation that the (conserved) super-Hamiltonian is

$$H \equiv p_\alpha \frac{dx^\alpha}{d\lambda} + p_l \frac{dl}{d\lambda} - L = 0 , \tag{6.18}$$

as expected. Now we resume our working, and choose $\lambda = s$ so $g_{\alpha\beta} u^\alpha u^\beta = 1$, to manipulate the equations of motion into the form

$$\frac{du^\mu}{ds}+\Gamma^\mu_{\beta\gamma}u^\beta u^\gamma=-g^{\mu\alpha}\frac{\partial g_{\alpha\beta}}{\partial l}\frac{dl}{ds}u^\beta+\left[\frac{1}{2\theta}\frac{d\theta}{ds}-\frac{2}{l}\frac{dl}{ds}\right]u^\mu \tag{6.19}$$

$$\frac{d^2l}{ds^2}-\left(\frac{1}{2\theta}\right)\frac{d\theta}{ds}\frac{dl}{ds}=-\frac{l}{L^2}-\frac{1}{2}\left(\frac{l^2}{L^2}\right)\frac{\partial g_{\beta\gamma}}{\partial l}u^\beta u^\gamma \quad . \tag{6.20}$$

Here $\Gamma^\mu_{\beta\gamma}$ is the usual 4D Christoffel symbol of the second kind. With the parameter along the path as 4D proper time, $\theta=\left(l^2/L^2\right)-\left(dl/ds\right)^2$, and the last equation can be written as

$$\frac{d^2l}{ds^2}-\frac{1}{l}\left(\frac{dl}{ds}\right)^2=-\left[\frac{l^2}{L^2}-\left(\frac{dl}{ds}\right)^2\right]\left[\frac{1}{l}+\frac{1}{2}\frac{\partial g_{\beta\gamma}}{\partial l}u^\beta u^\gamma\right]. \tag{6.21}$$

Substituting for θ in (6.19) and using (6.21), the 4D equations of motion can be written

$$\frac{du^\mu}{ds}+\Gamma^\mu_{\beta\gamma}u^\beta u^\gamma=F^\mu$$

$$F^\mu\equiv\left(-g^{\mu\alpha}+\frac{1}{2}\frac{dx^\mu}{ds}\frac{dx^\alpha}{ds}\right)\frac{dl}{ds}\frac{dx^\beta}{ds}\frac{\partial g_{\alpha\beta}}{\partial l} \quad . \tag{6.22}$$

The extra component of the equation of motion (6.21) can be rewritten as

$$\frac{d^2l}{ds^2}-\frac{2}{l}\left(\frac{dl}{ds}\right)^2+\frac{l}{L^2}=-\frac{1}{2}\left[\frac{l^2}{L^2}-\left(\frac{dl}{ds}\right)^2\right]\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}\frac{\partial g_{\alpha\beta}}{\partial l} \quad . \tag{6.23}$$

This completes the formal part of our analysis. Basically, if the coordinates and metric are put into the canonical form (6.8), the 4D equations of motion (6.22) show that ordinary geodesic motion is modified by the addition of a quantity F^μ which is a kind of force per unit inertial mass. (This is the analog in the canonical gauge of the extra terms we found in the electromagnetic gauge in Section 5.3.) The new force is nonzero if the 4D metric depends on the fifth coordinate $\left(\partial g_{\alpha\beta}/\partial l\neq 0\right)$ and there is motion in the fifth dimension $\left(dl/ds\neq 0\right)$. The rate of motion in the fifth dimension is given by a solution of (6.23) and is in general nonzero.

As a technical aside, we note that the latter equation can be once integrated. In (6.23), if we write $z \equiv 1/l$, $Z \equiv u^\alpha u^\beta \left(\partial g_{\alpha\beta} / \partial l\right)$ and put $L = 1$, that equation becomes

$$-\frac{1}{z^2}\frac{d^2 z}{ds^2}+\frac{1}{z}=-\frac{1}{2}\left[\frac{1}{z^2}-\frac{1}{z^4}\left(\frac{dz}{ds}\right)^2\right]Z\ .$$

Multiplying both sides of this by $-2z^2\ (dz/ds)$ gives

$$2\,\frac{dz}{ds}\left[\frac{d^2 z}{ds^2}-z\right]=\left[\left(\frac{dz}{ds}\right)^2-z^2\right]\left[-\frac{Z}{z^2}\,\frac{dz}{ds}\right],$$

which can be integrated to yield

$$\left(\frac{dz}{ds}\right)^2-z^2=C\exp\left[-\int_{z_0}^{z} Z\left(\frac{dz}{z^2}\right)\right]. \tag{6.24}$$

Here C is an arbitrary constant and $z_0 = z(s_0)$ is evaluated at some fiducial value of the 4D interval. For the special case $Z = 0$ when the 4D part of the metric does not depend on the extra coordinate, (6.24) is solved by $z = z_0 \cosh\,(s - s_0)$ with $C = -z_0^2$. In general, (6.24) gives on replacing the original variable the relation

$$\left(\frac{dl}{ds}\right)^2=l^2-\frac{l^4}{l_0^2}\exp\left[\int_{l_0}^{l} Z\,dl\right]. \tag{6.25}$$

This gives the square of the velocity in the extra dimension in terms of the 4D interval and a fiducial value of the extra coordinate $\left(dl/ds = 0 \text{ at } l = l_0\right)$, and is a useful relation.

Returning now to the 4D equations of motion (6.22), we note that the new force F^μ (per unit inertial mass) is necessarily connected to the existence of the fifth dimension. For from (6.22) we can form the scalar quantity

$$F^\mu u_\mu = -\frac{1}{2}u^\alpha u^\beta \frac{\partial g_{\alpha\beta}}{\partial l}\,\frac{dl}{ds}\ , \tag{6.26}$$

which has no 4D analog and brings out the dependencies noted above. Another way to see what is involved here is to start from the 4D scalar $u^\alpha u_\alpha = 1$, where we recall that in our approach this means $u^\alpha(s) g_{\alpha\beta}(x^\alpha, l) u^\beta(s) = 1$. The application of the total (4D covariant) derivative *D/ds* to this, along with the observation that $Dg_{\alpha\beta}/ds = \left(\partial g_{\alpha\beta}/\partial l\right)(dl/ds)$, results in

$$F^\alpha g_{\alpha\beta} u^\beta + u^\alpha u^\beta \frac{\partial g_{\alpha\beta}}{\partial l}\frac{dl}{ds} + u^\alpha g_{\alpha\beta} F^\beta = 0 \ . \tag{6.27}$$

This rearranged gives back (6.26). We see that the existence of the force (per unit inertial mass) F^μ, and its associated scalar $F^\mu u_\mu$ (which is a kind of power), follows from the fact that we have used the 4D interval s to characterize the motion in a 5D manifold. This makes sense from a practical point of view, because we have a body of 4D physics which we wish to interpret in terms of 5D geometry. However, it is important to realize that $F^\mu u_\mu \neq 0$ is a unique indication of the existence of at least one higher dimension.

To appreciate this, consider for the moment 4D Einstein gravity with other known forces. For example, the Lorentz force of electrodynamics obeys the relation $Du^\mu/ds = (q/m)F^\mu{}_\nu u^\nu$ where *q/m* is the charge/mass ratio of a test particle and $F^\mu{}_\nu$ is the (Faraday) field tensor. The antisymmetry of the latter means that $F_{\mu\nu}u^\mu u^\nu = 0$, so $F_\mu = (q/m)F_{\mu\nu}u^\nu$ obeys $F_\mu u^\mu = 0$. As another example, the force due to the pressure of a perfect fluid can be seen to obey a similar relation by considering the energy-momentum tensor $T^{\mu\nu} = (\rho + p)u^\mu u^\nu - pg^{\mu\nu}$. This obeys $T^{\mu\nu}{}_{;\nu} = 0$ or

$$(\rho + p)_{,\nu}\, u^\mu u^\nu + (\rho + p)u^\mu{}_{;\nu}u^\nu + (\rho + p)u^\mu u^\nu{}_{;\nu} - p_{,\nu}g^{\mu\nu} = 0 \ . \tag{6.28}$$

This can be multiplied by u_μ, and the term $(\rho + p)u^\mu{}_{;\nu}\, u^\nu u_\mu$ dropped, because by taking the covariant derivative of $u^\mu u_\mu = 1$, we have $u^\mu{}_{;\nu}\, u_\mu = 0$ (see above). Then there results

$$(\rho + p)u^\nu{}_{;\nu} = -\rho_{,\nu}u^\nu \tag{6.29}$$

which put back into (6.28) gives

$$F^\mu \equiv u^\mu{}_{;\nu}u^\nu = \frac{p_{,\nu}\left(g^{\mu\nu} - u^\mu u^\nu\right)}{(\rho + p)} \quad . \tag{6.30}$$

Again by virtue of $u^\mu u_\mu = 1$, this implies $F^\mu u_\mu = 0$. This result and the preceding one are not so much meant as a reminder of known physics, as illustrations of the fact that timelike motions in 4D general relativity with other known forces obey equations of motion of the form $Du^\mu / ds = F^\mu$ with $F_\mu u^\mu = 0$.

Conversely, the new force (6.22) derived in this section has $F_\mu u^\mu \neq 0$. Thus at least part of the new force (per unit inertial mass) F^μ of (6.22) must be non-4D in origin. To see this explicitly, we can express F^μ as the sum of a component $\left(N^\mu\right)$ normal to the 4-velocity u^μ of the particle and a component $\left(P^\mu\right)$ parallel to it. Thus $F^\mu = N^\mu + P^\mu$, where

$$N^\mu = \left(-g^{\mu\alpha} + u^\mu u^\alpha\right)u^\beta \frac{\partial g_{\alpha\beta}}{\partial l}\frac{dl}{ds} \tag{6.31}$$

$$P^\mu = -\frac{1}{2}u^\mu\left(u^\alpha u^\beta \frac{\partial g_{\alpha\beta}}{\partial l}\right)\frac{dl}{ds} \quad . \tag{6.32}$$

Clearly the normal component could be due to ordinary 4D forces (it obeys $N^\mu u_\mu = 0$ by construction), but there exists no 4D analog of the parallel component (it is $P^\mu = u^\mu F^\alpha u_\alpha$ and

has $P^\mu u_\mu \neq 0$ in general). This anomalous "fifth" force is therefore a consequence of the existence of the extra dimension.

6.5 Comments on the Fifth Force

In the preceding section, we saw that the components of the fifth force normal and parallel to the 4-velocity are both proportional to $\left(\partial g_{\alpha\beta} / \partial l\right)(dl/ds)u^\beta$. This is simply a consequence of employing the canonical coordinate system, and the fact that in this system dynamics is concentrated in the dependence of the spacetime metric on the fifth coordinate. However, it is useful to show that the form of the resulting equation for the fifth force (6.32) is general. We proceed to do this, and then make some comments on the fifth force in specific cases.

Let us consider a general 5D line element in the standard form for electromagnetism:

$$dS^2 = ds^2 - \Phi^2\left(dl + A_\mu\, dx^\mu\right)^2 \,. \tag{6.33}$$

Here $ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta$ is the spacetime metric as before, while Φ and A_μ are scalar and vector potentials. If $g_{\alpha\beta}$, Φ and A_μ are independent of the fifth coordinate l as in the traditional Kaluza-Klein theory, then using the results given in Chapter 5 it is straightforward to show that the force F^α (per unit inertial mass) on a particle is orthogonal to its 4-velocity u^α and there is no fifth force.

Imagine now a slight change of coordinates in 5D such that the new spacetime metric has the form

$$ds^2 = g_{\alpha\beta}\left(x^\mu, l\right)dx^\alpha dx^\beta \,. \tag{6.34}$$

The 5D geodesic remains invariant under the coordinate transformation. However, in 4D the force on the particle picks up an anomalous component parallel to the 4-velocity, while the normal component is expected to differ only slightly from F^α. To compute the new parallel component (fifth force), we simply differentiate

$$g_{\alpha\beta}\left(x^\mu, l\right)u^\alpha u^\beta = 1 \tag{6.35}$$

with respect to proper time s. Hence

$$g_{\alpha\beta,\gamma}\, u^\gamma u^\alpha u^\beta + \frac{\partial g_{\alpha\beta}}{\partial l}\frac{dl}{ds}\, u^\alpha u^\beta + 2\, g_{\alpha\mu}\frac{du^\mu}{ds}u^\alpha = 0 \ , \tag{6.36}$$

where the first term on the l.h.s. can be written as

$$\left(g_{\alpha\beta,\gamma} + g_{\alpha\gamma,\beta} - g_{\beta\gamma,\alpha}\right)u^\gamma u^\alpha u^\beta = 2 g_{\alpha\mu}\Gamma^\mu_{\beta\gamma}u^\gamma u^\alpha u^\beta \ , \tag{6.37}$$

using symmetries under the exchange of α and β. It follows that equation (6.36) can be expressed as

$$2 g_{\alpha\mu}u^\alpha\left(\frac{du^\mu}{ds} + \Gamma^\mu_{\beta\gamma}\, u^\beta u^\gamma\right) + \frac{\partial g_{\alpha\beta}}{\partial l}\frac{dl}{ds}\, u^\alpha u^\beta = 0 \ , \tag{6.38}$$

or using the definition of force per unit inertial mass

$$F^\mu u_\mu = -\frac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial l}\frac{dl}{ds}u^\alpha u^\beta \ . \tag{6.39}$$

Thus in the new perturbed coordinates a timelike fifth force appears which is given by

$$P^\mu = \left(-\frac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial l}\, u^\alpha u^\beta\right)\frac{dl}{ds}u^\mu \ . \tag{6.40}$$

This is the same expression as (6.32). However, the nature of the argument presented here implies that the fifth force has this form generally, regardless of the coordinate system employed.

Let us suppose that careful observations did indeed reveal a fifth force of the type described above. That is, suppose the net force on a particle were found to consist of an ordinary

component due to the four fundamental interactions, plus an anomalous timelike component proportional to the particle's four-velocity as in (6.40). Furthermore, let us suppose we insisted on interpreting these observations in terms of 4D physics (i.e., excluding a possible fifth dimension). Then the net force on the particle could be expressed as

$$\frac{D}{ds}\left(mu^{\alpha}\right) = m\left(N^{\alpha} + P^{\alpha}\right) . \tag{6.41}$$

Using $u_{\alpha}u^{\alpha} = 1$ and $u_{\alpha}Du^{\alpha}/ds = 0$ we then obtain

$$\frac{D}{ds}u^{\alpha} = N^{\alpha}$$

$$P^{\alpha} = \beta u^{\alpha} , \tag{6.42}$$

where $\beta \equiv m^{-1}\,dm/ds$. That is, the particle's mass would change with its proper time if we maintained that the world had only 4 dimensions when in fact it had 5. (This would presumably be a slow process, given the observational constraints alluded to in Sections 6.2 and 4.2.) Another related possibility, which we will come back to later, is to identify particle mass as the fifth coordinate. Then the extra component of the equation of motion (6.23) would give the rate of change of mass directly. In either case, we see that if there is a fifth dimension then it could show up as a variation in mass.

6.6 A Toy Model

The force F^{μ} of (6.22) and its associated scalar power (6.26) will exist for a wide class of solutions of the field equations $R_{AB} = 0$. What we wish to do here is to take the simplest model available and examine its consequences.

We start from the canonical metric (6.8) and put $L = 1$, $g_{\alpha\beta}(x^{\mu}, l) = l^{-2}\eta_{\alpha\beta}$. Then

$$dS^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta - dl^2 \tag{6.43}$$

and is flat. It might be useful to recall here that metrics which are 5D flat can correspond in certain systems of coordinates to metrics which are 4D curved, so we should not assume that (6.43) is devoid of physics. However, in the coordinates of (6.43), the geodesic equation obviously leads to $dx^A / dS =$ constants in terms of the 5D interval. However, in terms of the 4D interval, given by $ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta = l^{-2} \eta_{\alpha\beta} dx^\alpha dx^\beta$, we have $\eta_{\alpha\beta} u^\alpha u^\beta = l^2$ and the last component of the equations of motion (6.23) reads

$$\frac{d^2 l}{ds^2} - \frac{1}{l}\left(\frac{dl}{ds}\right)^2 = 0 \ . \tag{6.44}$$

Writing this as $d\left[l^{-1} dl / ds\right] / ds = 0$ we get

$$l = C_5 e^{\beta s} \ , \tag{6.45}$$

where C_5 is an arbitrary constant and $\beta \equiv l^{-1} dl / ds$ is a constant independent of the (4D) motion. [In more complicated situations, one may start with equation (6.25).] The first four components of the equations of motion (6.22) read

$$\frac{d^2 x^\mu}{ds^2} = F^\mu = \left(2l^2 \eta^{\mu\alpha} - \frac{dx^\mu}{ds}\frac{dx^\alpha}{ds}\right)\frac{dl}{ds}\frac{dx^\beta}{ds}\frac{\eta_{\alpha\beta}}{l^3}$$

$$= \frac{1}{l}\frac{dl}{ds}\frac{dx^\mu}{ds} = \beta \frac{dx^\mu}{ds} \ , \tag{6.46}$$

where β was just defined. If we express F^μ in terms of its components normal and parallel to the 4-velocity u^μ (see above), then $N^\mu = 0$ and $P^\mu = \beta u^\mu$. On performing a translation in 4D to produce an appropriate definition of the origin, and denoting 4 other arbitrary constants by C_μ, we have from (6.46) that

$$x^{\mu} = C_{\mu}\, e^{\beta s} \ . \tag{6.47}$$

So by (6.45) and (6.47), the motion in 5D in terms of the interval in 4D is summed up simply by $x^{A} = C_{A}\, e^{\beta s}$.

This solution depends on 5 arbitrary constants (C_A). But only 4 are independent, as the metric (6.43) with $U^{A} \equiv dx^{A} / dS$ implies the consistency relation $U^{A} U_{A} = 1$. To see what is involved here, let us consider the relationships between quantities in 5D, 4D and 3D when the 4D metric is $\eta_{\alpha\beta} = (+1, -1, -1, -1)$. We can define 3D velocities $\mathrm{v}^{i} \equiv dx^{i} / dt$ (i = 123) as usual, where by (6.47) we have

$$\mathrm{v}^{i} = \frac{dx^{i}}{ds}\frac{ds}{dt} = \frac{C_{i}}{C_{0}} \ . \tag{6.48}$$

The 4D interval $ds^{2} = l^{-2}\eta_{\alpha\beta} dx^{\alpha} dx^{\beta}$ then invites us to write

$$l^{2} = \eta_{\alpha\beta}\frac{dx^{\alpha}}{ds}\frac{dx^{\beta}}{ds} = \left(1 - \mathrm{v}^{2}\right)\left(\frac{dt}{ds}\right)^{2}$$

$$= \left(1 - \mathrm{v}^{2}\right)\left(C_{0}\beta e^{\beta s}\right)^{2} \quad = \quad \left(1 - \mathrm{v}^{2}\right)\left(C_{0}\beta\right)^{2}\left(\frac{l}{C_{5}}\right)^{2} \ . \tag{6.49}$$

Modulo the sign choice involved in taking the square root, we therefore have $C_{5} = C_{0}\beta\left(1 - \mathrm{v}^{2}\right)^{1/2}$. That is, $C_{5} = C_{0}\beta$ in the frame where $\mathrm{v} \equiv \left(\mathrm{v}^{i}\mathrm{v}_{i}\right)^{1/2}$ is zero. The 5D and 4D intervals (again modulo a choice of sign) are related via

$$dS = ds\left[l^{2} - \left(\frac{dl}{ds}\right)^{2}\right]^{1/2} = \left(1 - \beta^{2}\right)^{1/2} C_{5} e^{\beta s} ds \ , \tag{6.50}$$

where we have used (6.45). The velocities in terms of the 5D interval and the velocities in terms of the 4D interval are related, since $dx^A / dS = \left(dx^A / ds\right)\left(ds / dS\right)$. Using (6.45), (6.47) and (6.50) we have

$$U^A = \frac{dx^A}{dS} = \frac{\beta}{\left(1-\beta^2\right)^{1/2}} \frac{C_A}{C_5} \ . \tag{6.51}$$

This shows, as we stated above, that the 5-velocities obey $dx^A / dS =$ constants. Also, the consistency relation we stated at the beginning of this paragraph, namely $U^A U_A = 1$, now reads $C_5^2 = \beta^2\left(C_0^2 - C_1^2 - C_2^2 - C_3^2\right)$. This gives $C_5 = \beta C_0$ in the frame where v is zero, agreeing with what was noted above. The upshot of these considerations is that there is consistency between how we describe the motion in 3D, 4D and 5D; and that in the toy model we are looking at the motion in the extra dimension is such that $\beta = l^{-1} dl / ds$ is a constant.

In general, therefore, the 4D equation of motion (6.46) involves a force F^μ and a power $F^\mu u_\mu$ that are both nonzero:

$$F^\mu = \beta u^\mu$$

$$F^\mu u_\mu = \beta \ . \tag{6.52}$$

The value of the constant β is not determined by the toy model being used here, and neither is its sign. These things require the use of more sophisticated models.

6.7 Conclusion

Gauges in 5D field theory control to a certain extent what physical insights ensue, and in this chapter we have concentrated on the canonical one (6.8) which brings out effects of the extra coordinate. For this gauge, the 5D field equations show that while the cosmological constant

may be identified by comparison with the 4D equations of general relativity, this parameter does not fit naturally into the induced-matter picture, being replaced by the more general concept of properties for the vacuum which are determined locally. The canonical gauge has major implications for dynamics. It defines the Lagrangian (6.12) which leads to equations of motion in 4D spacetime (6.22) and the extra dimension (6.23). The former involve a modification to the usual 4D motion of a test particle which has the nature of a new force (per unit inertial mass). This is nonzero if the metric depends on the extra coordinate (which in general it will if there is induced matter present) and if there is movement in the extra dimension (which in general there will be since we have no reason to presume things are static there and no gauge freedom left to impose this). The total force F^μ of (6.22) has a velocity-contracted scalar associated with it, namely $F^\mu u_\mu$ of (6.26), which is a kind of power and is nonzero under the noted conditions. This is an indicator of the existence of at least one extra dimension, since known (4D) interactions involve $F^\mu u_\mu = 0$. The total force F^μ can be expressed in terms of the component normal to the 4-velocity N^μ and the component parallel to it P^μ, as in (6.31) and (6.32). The former obeys $N^\mu u_\mu = 0$, so strictly speaking it is the fact that $P^\mu u_\mu \neq 0$ which would indicate the existence of (at least) a fifth dimension. It is actually a general property of x^4-dependent metrics in 5D theory that there appears a new timelike or fifth force P^μ (6.40). If, however, one tried to interpret this from a 4D perspective, it could manifest itself as a time variation in particle mass as in (6.42). A similar relation (6.52) results when one calculates the rate of change of the extra coordinate in a toy model. Both variations are of course constrained by data on dynamics, and in particular by limits on the time variation of the gravitational interaction, to which we have alluded earlier in this chapter and in previous ones.

Given this, two questions occur. First, why has the fifth force escaped detection by traditional dynamical tests such as those applied in the solar system? Second, what new ways are available to look for it?

References (Chapter 6)

Bekenstein, J.D. 1977, Phys. Rev. D 15, 1458.

Hawking, S.W., Ellis, G.F.R. 1973, The Large-Scale Structure of Space-Time (Cambridge U. Press, Cambridge), 44.

Henriksen, R.N., Emslie, A.G., Wesson, P.S. 1983, Phys. Rev. D 27, 1219.

Jordan, P. 1955, Schwerkraft und Weltall (Vieweg, Braunschweig).

Leibbrandt, G. 1994, Non-Covariant Gauges (World Scientific, Singapore).

Liu, H., Mashhoon, B. 1995, Ann. Phys. (Leipzig) 4, 565.

Mashhoon, B., Liu, H., Wesson, P.S. 1994, Phys. Lett. B 331, 305.

Mashhoon, B., Wesson, P.S., Liu, H. 1998, Gen. Rel. Grav., 30, 555.

Misner, C., Thorne, K., Wheeler, J.A. 1973, Gravitation (Freeman, San Francisco), 506.

Myers, R.C., Perry, M.J. 1986, Ann. Phys. (New York) 172, 304.

Ng, Y.J. 1992, Int. J. Mod. Phys. D1, 145.

Sajko, W.N., Wesson, P.S., Liu, H. 1998, J. Math. Phys. 39, 2193.

Shapiro, I.I. 1993 in Quantum Gravity and Beyond – Essays in Honour of Louis Witten (eds., Mansouri, F., Scanio, J.J., World Scientific, Singapore), 180.

Wald, R. 1984, General Relativity (U. of Chicago Press, Chicago), 255.

Weinberg, S. 1972, Gravitation and Cosmology (Wiley, New York), 162, 253.

Weinberg, S. 1989, Rev. Mod. Phys. 61, 1.

Wesson, P.S. 1978, Cosmology and Geophysics (Oxford U. Press-Hilger, New York).

Wesson, P.S. 1997, Int. J. Mod. Phys. D 6, 643.

Will, C.M. 1992, Int. J. Mod. Phys. D 1, 13.

7. CANONICAL SOLUTIONS AND PHYSICAL QUANTITIES

"General relativity will eventually overtake the rest of physics"

(George Ellis, Waterloo, 1997)

7.1 Introduction

In Chapter 6 we found that when the 5D metric is written in the canonical form, where the extra dimension is explicit, the equations of motion yield in a relatively straightforward way a fifth force which is unlike those of classical electromagnetism or 4D general relativity. However, this force does not manifest itself in the solar system at the level of order 0.1% set by the classical tests. It also does not appear to manifest itself in cosmological systems, though the level here is less stringent due to observational uncertainties. In the present chapter we will find that there is a simple yet fundamental reason for this: canonical solutions exist in which the fifth force is zero. That is, there are exact solutions of the 5D field equations which possess a simple symmetry and result in the same dynamics as 4D general relativity.

This means that we have to look elsewhere for the effects of the fifth dimension. The best option for detection would appear to be a gyroscope experiment in Earth orbit. We discussed this before in Chapter 3 for the soliton metric, but we revisit the problem here for the 1-body metric in canonical form and give a detailed analysis. We will find that while the orbit of the gyroscope is the same as in general relativity, the precession of its spin axis is different.

We will then return to, and make concrete, an issue which has been considerably discussed in the literature and which we have mentioned at several places in this account, namely the connection between the extra coordinate x^4 and the rest mass of a test particle m. This is a manifestation of Mach's principle (Chapter 1); and for the induced-matter interpretation of 5D

relativity, we know that when the metric is written in a certain form, dependency on x^4 generally implies matter consisting of particles with finite rest masses, while independency implies matter consisting of photons (Chapter 2). A connection between x^4 and m has significant implications for cosmology (Chapter 4); and in terms of the fifth force, we know that if it were interpreted in terms of 4D physics it would show up as a small variation in the mass of a test particle (Chapter 6). In the present chapter, we will show that there is a particular coordinate system where in fact we can write just $x^4 = m$. In the same system, we will also find that the rate of change of x^4 measures the electric charge q.

Particle properties such as these are not normally thought of as being derivable from classical field theory, but are instead commonly regarded as the purview of quantum theory. It is true that a possible connection between these subjects has been discussed in terms of ND Kaluza-Klein theory with compactification and cylindricity, but this approach runs into well-known problems (see Chapter 1). We therefore choose to round out this chapter with a short speculation on how the induced-matter approach may be extended to particle physics.

7.2 The Canonical 1-Body Solution

The 1-body soliton solution (3.4) has been much discussed in the literature, and cannot be ruled out by solar system tests (Kalligas, Wesson and Everitt 1995). However, in the absence of 5D uniqueness theorems we should investigate the possibility that the field outside an object like the Sun might be described by a different solution.

Consider therefore the static, (3D) spherically-symmetric metric

$$dS^2 = \frac{l^2}{L_1^2}\left\{\left[1-\frac{2M}{r}-\frac{r^2}{L_2^2}\right]dt^2 - \left[1-\frac{2M}{r}-\frac{r^2}{L_2^2}\right]^{-1} dr^2 - r^2\left(d\theta^2+\sin^2\theta d\phi^2\right)\right\} - dl^2 . \tag{7.1}$$

Here M is a constant associated with the mass at the centre of the 3-geometry, and L_1 and L_2 are constants with the dimensions of lengths (so t, r, l have conventional dimensions). The non-zero components of R_{AB} for (7.1) are

$$R_{00} = \frac{3\left(rL_2^2 - 2ML_2^2 - r^3\right)\left(L_1^2 - L_2^2\right)}{r\,L_1^2\,L_2^4}$$

$$R_{11} = \frac{-3r\left(L_1^2 - L_2^2\right)}{L_1^2\left(r\,L_2^2 - 2ML_2^2 - r^3\right)}$$

$$R_{22} = \frac{-3r^2\left(L_1^2 - L_2^2\right)}{L_1^2 L_2^2}$$

$$R_{33} = \left(\sin^2\theta\right)R_{22} \quad . \tag{7.2}$$

We see that the field equations $R_{AB} = 0$ are only satisfied if $L_1 = L_2 = L$ (say). Writing $L^2 = 3/\Lambda$ where Λ will be identified with the cosmological constant, (7.1) becomes

$$dS^2 = \frac{\Lambda l^2}{3}\left\{\left[1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}\right]dt^2 - \left[1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}\right]^{-1} dr^2 - r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right\} - dl^2 \ . \tag{7.3}$$

The part of this metric inside the curly brackets is the 4D Schwarzschild-de Sitter one (see Section 6.3 for an alternative form). The canonical solution (7.3) is quite different from the soliton solution, and a natural question is how the new form fares in comparison with the dynamical tests in the solar system.

Geodesic motion was studied in the preceding chapter for metrics with the canonical form

$$dS^2 = \gamma_{AB}dx^A dx^B = \left(\frac{l^2}{L^2}\right)g_{\alpha\beta}dx^\alpha dx^\beta - dl^2 = \left(\frac{l^2}{L^2}\right)ds^2 - dl^2 \ ,$$

and the results can conveniently be summarized here in the relations

$$\frac{d^2x^\mu}{ds^2}+\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}=F^\mu$$

$$F^\mu\equiv\left(-g^{\mu\alpha}+\frac{1}{2}\frac{dx^\mu}{ds}\frac{dx^\alpha}{ds}\right)\frac{dl}{ds}\frac{dx^\beta}{ds}\frac{\partial g_{\alpha\beta}}{\partial l}$$

$$\frac{d^2l}{ds^2}-\frac{2}{l}\left(\frac{dl}{ds}\right)^2+\frac{l}{L^2}=-\frac{1}{2}\left[\frac{l^2}{L^2}-\left(\frac{dl}{ds}\right)^2\right]\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}\frac{\partial g_{\alpha\beta}}{\partial l}\quad . \tag{7.4}$$

Here F^μ is the fifth force (per unit mass) which manifests itself in 4D due to the existence of the fifth dimension. But for the solution (7.3), $\partial g_{\alpha\beta}/\partial l=0$ so $F^\mu=0$, and the 4D motion is described by the same 4D geodesic equation as in Einstein's theory. The rather remarkable result follows that there is no way to tell by the classical tests of relativity if the solar system is described by the 5D metric (7.3) or only its 4D part.

7.3 The Canonical Inflationary Solution

For cosmology, a similar situation exists though it requires a bit more comment. In standard 4D cosmology, the universe is described by solutions whose 5D analogs are the $l-$dependent Ponce de Leon metrics we studied in Chapters 2 and 4. However, we saw in the latter place that the recovery of standard 4D results such as Hubble's law depends on a choice of 5D coordinates, all of which are actually arbitrary. This makes it problematical to decide by astrophysics if the universe is 5D or 4D in nature. Also, to resolve problems to do with horizons in standard 4D cosmology, it is usual to appeal to an era of inflationary or exponential growth at very early times. In this regard, it turns out that we encounter the same situation as in the 1-body problem above.

To see this, consider the (3D) homogeneous, isotropic metric

$$dS^2 = \frac{l^2}{L_1^2}\left\{dt^2 - e^{t/L_2}\left[dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right]\right\} - dl^2 \quad . \tag{7.5}$$

The non-zero components of R_{AB} for this are

$$R_{00} = \frac{3(L_1^2 - 4L_2^2)}{4L_1^2 L_2^2}$$

$$R_{11} = \frac{-3e^{t/L_2}(L_1^2 - 4L_2^2)}{4L_1^2 L_2^2}$$

$$R_{22} = \frac{-3r^2 e^{t/L_2}(L_1^2 - 4L_2^2)}{4L_1^2 L_2^2}$$

$$R_{33} = \left(\sin^2\theta\right)R_{22} \quad . \tag{7.6}$$

We see that the field equations are only satisfied if $L_1 = 2L_2 = L$ (say). Writing again $L^2 = 3/\Lambda$, (7.5) becomes

$$dS^2 = \frac{\Lambda l^2}{3}\left\{dt^2 - \exp\left(2\sqrt{\Lambda/3}\,t\right)\left[dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right]\right\} - dl^2 . \tag{7.7}$$

The part of this metric inside the curly brackets is the 4D space-flat de Sitter one (see section 6.3 for an alternative form). The 5D solution (7.7) has $\partial g_{\alpha\beta}/\partial l = 0$ so $F^\mu = 0$ by (7.4). It follows that there is no way using conventional dynamics to tell if inflationary cosmology is described by the 5D metric (7.7) or only its 4D part.

The im(possibility) of detecting a fifth dimension for canonical solutions has been discussed by Wesson, Mashhoon and Liu (1997). The detection of effects due to $x^4 = l$ may be out of reach using conventional dynamical methods, but alternatives exist. For example, both the 1-body solution (7.3) and the inflationary solution (7.7) have perforce $\Lambda \neq 0$, which in 4D notation corresponds to a density and pressure for the vacuum given by $\rho = -p = \Lambda/8\pi$ (see

Chapters 1 and 6). This may be detectable thermodynamically. As another option, an investigation of the relevant equations shows that the degeneracy between the 4D and 5D dynamics of a simple test particle is lifted if the latter has spin. From both the astrophysical and experimental sides, this seems to us to be the best option available for investigating the existence of a canonical fifth dimension.

7.4 A Spinning Object in a 5D Space

The motion of a spinning object such as an elementary particle or satellite in a 5D manifold that extends Schwarzschild spacetime is a problem with widespread applications. In what follows, we therefore eschew adherence to any particular physical interpretation of 5D relativity, and follow Liu and Wesson (1996) in regarding this problem as a mathematical exercise in higher-dimensional dynamics.

We start with the canonical, 1-body solution to $R_{AB} = 0$ (7.3) in the form

$$dS^2 = \frac{l^2}{L^2}\left[\left(1-\frac{2M}{r}-\frac{r^2}{L^2}\right)dt^2 - \left(1-\frac{2M}{r}-\frac{r^2}{L^2}\right)^{-1} dr^2 - r^2\left(d\theta^2+\sin^2\theta d\phi^2\right)\right] - dl^2 \ . \tag{7.8}$$

Here L and M are constants. The 5D Christoffel symbols of the second kind for (7.8) are:

$$\Gamma^l_{tt} = -\frac{l\left(r^3 - rL^2 + 2ML^2\right)}{rL^4}$$

$$\Gamma^l_{rr} = -\frac{lr}{\left(r^3 - rL^2 + 2ML^2\right)} \qquad \Gamma^l_{\theta\theta} = -\frac{lr^2}{L^2}$$

$$\Gamma^l_{\phi\phi} = \frac{lr^2(\cos 2\theta - 1)}{2L^2}$$

$$\Gamma^t_{tr} = -\frac{\left(r^3 - ML^2\right)}{r\left(r^3 - rL^2 + 2ML^2\right)} \qquad \Gamma^t_{tl} = -\frac{1}{l}$$

$$\Gamma^r_{tt} = \frac{\left(r^3 - rL^2 + 2ML^2\right)\left(r^3 - ML^2\right)}{r^3 L^4}$$

$$\Gamma^r_{rr} = -\frac{\left(r^3 - ML^2\right)}{r\left(r^3 - rL^2 + 2ML^2\right)} \qquad \Gamma^r_{rl} = \frac{1}{l}$$

$$\Gamma^r_{\theta\theta} = \frac{\left(r^3 - rL^2 + 2ML^2\right)}{L^2}$$

$$\Gamma^r_{\phi\phi} = -\frac{(\cos 2\theta - 1)\left(r^3 - rL^2 + 2ML^2\right)}{2L^2}$$

$$\Gamma^\theta_{r\theta} = \frac{1}{r} \qquad \Gamma^\theta_{\theta l} = \frac{1}{l} \qquad \Gamma^\theta_{\phi\phi} = -\frac{\sin 2\theta}{2}$$

$$\Gamma^\phi_{r\phi} = \frac{1}{r} \qquad \Gamma^\phi_{\theta\phi} = \frac{\cos\theta}{\sin\theta} \qquad \Gamma^\phi_{\phi l} = \frac{1}{l}\,. \tag{7.9}$$

We will need these in order to carry out the rest of the analysis.

An object which moves on an orbit with a 5-velocity $U^A \equiv dx^A/dS$ and has an intrinsic spin vector S^A is governed by equations that reflect geodesic motion and relations set by the metric (Kalligas, Wesson and Everitt 1995). For the orbit and spin, respectively, we have to solve

$$\frac{dU^A}{dS} + \Gamma^A_{BC} U^B U^C = 0 \qquad g_{BC} U^B U^C = 1 \tag{7.10}$$

$$\frac{dS^A}{dS} + \Gamma^A_{BC} S^B U^C = 0 \qquad g_{BC} S^B U^C = 0\,. \tag{7.11}$$

The orbit we choose to be circular and to lie in the plane $\theta = \pi/2$, so $U^1 = U^r = 0$ and $U^2 = U^\theta = 0$. The remaining 3 components of the 5-velocity may be verified by substitution into (7.10) to be given by

$$U^0 = \frac{dt}{dS} = \frac{1}{\sqrt{1-3M/r}} \frac{Ll_0}{l^2}$$

$$U^3 = \frac{d\phi}{dS} = \frac{\sqrt{M/r - r^2/L^2}}{\sqrt{1-3M/r}} \frac{Ll_0}{l^2 r}$$

$$U^4 = \frac{dl}{dS} = -\frac{S}{l} \,. \tag{7.12}$$

Here l_0 is an arbitrary constant. The spin we leave unrestricted for now, and its 5 components may be verified by a lengthy substitution into (7.11) to be given by

$$S^0 = \frac{A_0}{l} \sin\left[\phi_0 - \Omega(s - s_0)\right] - \frac{B_0 S}{l^2}$$

$$S^1 = \frac{A_1}{l} \cos\left[\phi_0 - \Omega(s - s_0)\right]$$

$$S^2 = \frac{B_2}{l}$$

$$S^3 = \frac{A_3}{l} \sin\left[\phi_0 - \Omega(s - s_0)\right] - \frac{B_3 S}{l^2}$$

$$S^4 = \frac{B_4}{l} \,. \tag{7.13}$$

Here ϕ_0 and s_0 are arbitrary constants, and one should be careful to distinguish between the 5D scalar interval S, the 5D components of the spin vector S^A and the 4D scalar interval (or proper time) s. Also in (7.13), we have introduced functions A_0, A_1, A_3 and B_0, B_2, B_3, B_4 which depend on r but on only 3 constants (say H_1, H_2, H_4):

$$A_0 = \frac{\sqrt{M/r - r^2/L^2}}{\sqrt{1-2M/r - r^2/L^2}} \frac{H_1 L}{\sqrt{1-3M/r}}$$

$$A_1 = \sqrt{1-2M/r - r^2/L^2}\, H_1 L$$

$$A_3 = \frac{\sqrt{1-2M/r-r^2/L^2}}{r\sqrt{1-3M/r}}\, H_1 L$$

$$B_0 = \frac{H_4 L}{\sqrt{1-3M/r}}$$

$$B_2 = \frac{H_2 L}{r}$$

$$B_3 = \frac{\sqrt{M/r-r^2/L^2}}{r\sqrt{1-3M/r}}\, H_4 L$$

$$B_4 = H_4 l_0 \quad . \tag{7.14}$$

The constants H_1, H_2, H_4 are arbitrary but normalized amplitudes of the spin S^A, since the components (7.13) satisfy

$$S_A S^A = -\left(H_1^2 + H_2^2 + H_4^2\right) \quad . \tag{7.15}$$

By comparison, the last parameter Ω which figures in (7.13) is not arbitrary, and is in fact the spin angular velocity of the particle in its orbit:

$$\Omega = \frac{1}{r}\sqrt{\frac{M}{r} - \frac{r^2}{L^2}} \quad . \tag{7.16}$$

It should be noted that the latter is not the same as the orbital angular velocity, which is defined by

$$\omega \equiv \frac{d\phi}{ds} = \frac{1}{r}\frac{\sqrt{M/r-r^2/L^2}}{\sqrt{1-3M/r}} \quad . \tag{7.17}$$

The two are related by

$$\omega = \frac{\Omega}{\sqrt{1-3M/r}} \quad , \tag{7.18}$$

a discrepancy which results in a (geodetic) precession of the spin axis of an object in orbit around a central mass even in 4D. We looked at this discrepancy for the soliton metric in Chapter 3, and here we proceed to look at it for the canonical metric (7.8) in a reasonably physical situation.

Let us assume that a spinning object is put into circular orbit around a central mass so that its spin vector lies in the plane of the orbit. Then $S^2 = 0$. Let us also assume that by mechanical means we have $rS^3 << S^1$ (due to our choice of coordinates S^0, S^1 and S^4 are dimensionless while S^2 and S^3 have dimension of inverse length, hence this physical condition). We make observations in 4D, so by analogy with what we did in previous chapters for the 4-velocities, we can define $S_s^A = S^A (dS/ds)$ where $dS/ds = l^2/Ll_0$ by previous relations. Then by (7.13) and (7.14), the components of S_s^A are given by:

$$S_s^0 = \frac{H_1 l}{l_0} \frac{\sqrt{M/r - r^2/L^2}}{\sqrt{(1-2M/r-r^2/L^2)(1-3M/r)}} \sin\left[\phi_0 - \Omega(s-s_0)\right] - \frac{H_4 S}{l_0\sqrt{1-3M/r}}$$

$$S_s^1 = \frac{H_1 l}{l_0}\sqrt{1-\frac{2M}{r}-\frac{r^2}{L^2}}\cos\left[\phi_0 - \Omega(s-s_0)\right]$$

$$S_s^2 = \frac{H_2 l}{l_0 r} = 0$$

$$S_s^3 = \frac{H_1 l}{l_0 r}\frac{\sqrt{1-2M/r-r^2/L^2}}{\sqrt{(1-3M/r)}}\sin\left[\phi_0 - \Omega(s-s_0)\right] - \frac{H_4 S}{l_0 r}\frac{\sqrt{M/r-r^2/L^2}}{\sqrt{1-3M/r}}$$

$$S_s^4 = \frac{H_4 l}{L} \quad . \tag{7.19}$$

At the start of an orbit we set $s = s_0$, $S = S_0$, $\phi_0 = 0$. At the end of an orbit $s = s_0 + \Delta s$, $S = S_0 + \Delta S$ where $\Delta s = 2\pi/\omega = 2\pi r\sqrt{1-3M/r}\Big/\sqrt{M/r-r^2/L^2}$ from (7.17) and

$\Delta S = \left(l^2 / Ll_0\right)\Delta s$ from above. The angle ϕ does not go exactly to 2π, but by analogy with the 4D problem has an offset $\delta\phi = r\Delta S_s^3 / S_s^1$ where $\Delta S_s^3 \equiv S_s^3(s) - S_s^3(s_0)$. These relations combined with (7.19) in the weak-field approximation $\left(r^2 / L^2 << M / r << 1\right)$ give the precession angle for one orbit, and the effect will accumulate over the $1/\Delta s \cong \sqrt{M / 4\pi^2 r^3}$ orbits in a year (Liu and Wesson 1996). For an experiment of short duration $\left(\Delta s << L\right)$, the expression for the precession per orbit is fairly simple:

$$\delta\phi = \frac{3\pi M}{r} - 2\pi\left(\frac{H_4}{H_1}\right)\left(\frac{r}{L}\right) \quad . \tag{7.20}$$

This is the analog of (3.66) for the soliton metric (3.4), but now for the different 1-body canonical metric (7.8). The first term in (7.20) is the usual geodetic precession found in 4D general relativity. The second term will be small because it depends on the (presumably cosmological) length L in the metric. However, the terms in (7.20) depend quite differently on the radial distance r, so in principle they could be separated. In fact, the conditions we have used to derive (7.20) might be realized in an Earth/artificial satellite situation as envisaged in the GP-B project (see Jantzen, Keiser and Ruffini 1996). Alternatively, the conditions we have used are close to those in the Sun/Uranus situation. In these and other situations, it is in principle possible to differentiate between 4D and 5D manifolds by measuring the contribution to the precession in 4D due to spin in the fifth dimension.

7.5 The Nature of Mass and Charge

It is widely acknowledged that the best way to test classical field theories of gravity and electromagnetism is through the dynamics of a test particle of rest mass m and charge q. This view has some historical justification insofar as it validates general relativity and classical

electromagnetism. But it is flawed, because in both theories there is no physical explanation for m and q on the same basis as there is for the other quantities which appear in the equations. This problem was mentioned in Chapter 1, and there have been several notable attempts to tackle it (Jammer 1961; Hoyle and Narlikar 1974; Barbour and Pfister 1995). A partial resolution was obtained in Chapter 5, where we used the 4D part of the 5D geodesic equation to give a geometrical interpretation for the ratio q/m. We now wish to revisit this problem from a different angle (Wesson and Liu 1997), and use the canonical metric to offer a complete explanation of charge and mass.

Let us start by reconsidering dynamics in 4D and 5D, assuming that the former has to be embedded in the latter, but using only the metric and neglecting constraints that might derive from a set of field equations.

In 4D, we have a line element which is given in terms of a metric tensor by $ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta$, and a Lagrangian which is commonly defined as

$$L = m\frac{ds}{d\lambda} + q\, A_\mu \frac{dx^\mu}{d\lambda} \quad . \tag{7.21}$$

Here λ is a parameter and A_μ is the electromagnetic potential. This is acceptable, but clearly m and q are introduced ad hoc. Overlooking this for the moment, we form the momentum 4-vector

$$p_\alpha = \frac{\partial L}{\partial\left(dx^\alpha / d\lambda\right)} = m\, g_{\alpha\beta}\frac{dx^\beta}{ds} + q\, A_\alpha \quad . \tag{7.22}$$

That is, $p_\alpha = m g_{\alpha\beta}\, u^\beta + q A_\alpha$ and $p^\alpha \equiv g^{\alpha\beta} p_\beta = m u^\alpha + q A^\alpha$ where $u^\alpha \equiv dx^\alpha / ds$ is the 4-velocity. The product is

$$p_\alpha p^\alpha = m^2 u_\alpha u^\alpha + 2mq\, A_\alpha u^\alpha + q^2 A_\alpha A^\alpha \quad , \tag{7.23}$$

which is just $+m^2$ when we choose the signature to be $g_{\alpha\beta} = (+1,-1,-1,-1)$ and put $q = 0$. Locally $g_{\alpha\beta} \cong \eta_{\alpha\beta}$, and for a charged particle moving with a 3-velocity $v^{123} = dx^{123}/dt$ we have

$$p^0 = m\frac{dt}{ds} + qA^0 = \frac{m}{\left(1-v^2\right)^{1/2}} + qA^0 \tag{7.24}$$

$$p^{123} = m\frac{dx^{123}}{ds} + qA^{123} = \frac{mv^{123}}{\left(1-v^2\right)^{1/2}} + qA^{123}, \tag{7.25}$$

which we call the energy and (3D) momenta. These relations lead of course to the result $E^2 = p^2 + m^2$, which is the basis of particle physics and is in excellent agreement with experimental data. However, as pointed out above, there is no explanation of m or (if it is finite) of q.

In 5D, we have a line element which is given in terms of a metric tensor by $dS^2 = g_{AB}dx^A dx^B$ and which as in Chapter 5 we can rewrite as the sum of a 4D part and an extra part that depends on scalar and vector fields:

$$dS^2 = g_{\alpha\beta}dx^\alpha dx^\beta - \Phi^2\left(dl + A_\alpha dx^\alpha\right)^2. \tag{7.26}$$

The corresponding Lagrangian is

$$L \equiv m\frac{dS}{d\lambda} = m\left[g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} - \Phi^2\left(\frac{dl}{d\lambda} + A_\alpha\frac{dx^\alpha}{d\lambda}\right)^2\right]^{1/2}. \tag{7.27}$$

Here m is still introduced ad hoc, but the theory has one extra parameter associated with the fifth coordinate l and the charge q does not appear explicitly (see below). From (7.27) we can form the 4-part of the momentum 5-vector:

$$P_\alpha = \frac{\partial L}{\partial\left(dx^\alpha / d\lambda\right)} = m\left[g_{\alpha\beta}\frac{dx^\beta}{dS} + nA_\alpha\right]$$

$$n \equiv -\Phi^2\left(\frac{dl}{dS} + A_\alpha\frac{dx^\alpha}{dS}\right). \tag{7.28}$$

This is the 5D analog of the 4D relation (7.22). The extra or fifth component is

$$P_4 = mn\ . \tag{7.29}$$

We see that the scalar function n of (7.28) specifies the momentum in the extra dimension (see Chapters 1 and 5). We might expect n to be small in many situations, because from (7.26) and (7.28) we have $ds/dS = \left(1+n^2/\Phi^2\right)^{1/2}$ as in (5.22), so $ds \cong dS$ in this case. To get the contravariant forms of the covariant vectors (7.28) and (7.29), we note that

$$g^{AB} = \begin{bmatrix} g^{\alpha\beta} & -A^\alpha \\ -A^\alpha & \left(A^\mu A_\mu - \Phi^{-2}\right) \end{bmatrix}, \tag{7.30}$$

with $A^\mu \equiv g^{\mu\nu}A_\nu$. This gives $P^\alpha \equiv g^{\alpha B}P_B = mU^\alpha$ and $P^4 \equiv g^{4B}P_B = mU^4$ where $U^A \equiv dx^A/dS$ is the 5-velocity. The product is

$$P_A P^A = m^2\ . \tag{7.31}$$

This is the 5D analog of the 4D relation (7.23). However, q does not appear in the former whereas it does in the latter, so we need to inquire where this parameter fits into the extended geometry. To answer this, we rewrite (7.28) as

$$P_\alpha = m\left[g_{\alpha\beta}U^\beta + nA_\alpha\right]. \tag{7.32}$$

Although it is the 4-part of a 5-vector, the metric (7.26) from which it is derived is invariant under the 4D subset of general 5D coordinate transformations $x^\alpha \to \bar{x}^\alpha(x^\beta)$, and so therefore

must (7.32) be. That is, P_α is a 4-vector. To recast it in 4D form, we use from above $ds/dS = \left(1+n^2/\Phi^2\right)^{1/2}$ and find that

$$P_\alpha = \left[m\left(g_{\alpha\beta}u^\beta + \frac{n}{\left(1+n^2\Phi^{-2}\right)^{1/2}} A_\alpha \right) \right] \frac{ds}{dS} \quad . \tag{7.33}$$

This can be compared with the purely 4D relation (7.22):

$$p_\alpha = m\left(g_{\alpha\beta}\, u^\beta + \frac{q}{m} A_\alpha \right) . \tag{7.34}$$

Clearly the vectors are related via $P_\alpha = p_\alpha (ds/dS)$ and the charge/mass ratio of the test particle is

$$\frac{q}{m} = \frac{n}{\left(1+n^2\Phi^{-2}\right)^{1/2}} \quad . \tag{7.35}$$

This means that charge is equal to mass times a function of coordinates, so at this point we are half way to explaining both parameters in terms of geometry.

The result (7.35) derived here from the Lagrangian agrees with (5.29) of Chapter 5 derived from the 4D part of the 5D geodesic. The last was studied extensively in Section 5.3, and to speed the present analysis we recall that the motion in the $x^4 = l$ and $x^0 = t$ directions can be summed up in terms of two relations:

$$\frac{dn}{dS} = \frac{1}{2}\frac{\partial g_{AB}}{\partial l}\frac{dx^A}{dS}\frac{dx^B}{dS} \qquad n \equiv -\Phi^2\left(\frac{dl}{dS} + A_\alpha \frac{dx^\alpha}{dS}\right) \tag{7.36}$$

$$\frac{de}{dS} = \frac{1}{2}\frac{\partial g_{AB}}{\partial t}\frac{dx^A}{ds}\frac{dx^B}{dS} \qquad e \equiv \frac{g_{00}^{1/2}}{\left(1-\mathrm{v}^2\right)^{1/2}}\left(1+\frac{n^2}{\Phi^2}\right)^{1/2} + nA_0 \, . \tag{7.37}$$

The functions n,e are constants of the motion if the metric is independent of l,t respectively; and as we noted before the second is strongly reminiscent of the expression for the energy of a test particle in a static 4D spacetime:

$$\in = \frac{g_{00}^{1/2}}{\left(1-\mathrm{v}^2\right)^{1/2}} m + q\,A_0 \ . \tag{7.38}$$

The route to explaining charge and mass as geometrical objects is now apparent, insofar as we expect the q, m of 4D to be related to the n,e of 5D, subject to compatibility with (7.35) for the ratio.

Let us look at the mass first, for the case of an uncharged test particle ($q,\ n \to 0$). In (7.37), g_{00} refers to a 5D metric coefficient, and we take the canonical 5D metric in the weak-field case where the 4D part is close to flat, so $dS^2 \cong (l/L)^2 ds^2 - dl^2$ with $ds^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta$ and $\eta_{\alpha\beta} = \left(+1, -1, -1, -1\right)$, so $g_{00}^{1/2} = l/L$ in (7.37). In (7.38), g_{00} refers to a 4D metric coefficient, and in the corresponding limit $g_{00}^{1/2} = 1$. Then (7.37) and (7.38) read respectively

$$e = \frac{l}{L(1-\mathrm{v}^2)^{1/2}} \ , \qquad\qquad \in = \frac{m}{\left(1-\mathrm{v}^2\right)^{1/2}} \ . \tag{7.39}$$

Here e is dimensionless while $\in$ has the dimensions of a length (due to the ad hoc introduction of m in 4D). Thus we make the identification

$$\in \equiv eL = \frac{l}{\left(1-\mathrm{v}^2\right)^{1/2}} \ . \tag{7.40}$$

This means that the role of the 4D uncharged rest mass is played in 5D geometry by the extra coordinate.

Now let us look at the charge. A comparison of (7.36), (7.37) and (7.38) shows that we have to make a distinction between the mass of an uncharged particle (which we have above

labeled m) and the mass of a charged particle (which we here label m_q). The appropriate definitions are

$$m_q \equiv l\left(1+\frac{n^2}{\Phi^2}\right)^{1/2}, \qquad q \equiv nl \quad . \tag{7.41}$$

These mean that the mass is augmented by the electromagnetic self-energy of the particle, as one might expect; but more importantly that the role of 4D charge is played in 5D geometry by the extra momentum.

The ratio q/m_q derived from (7.41) agrees with (7.35), and in fact a more detailed investigation of the relations we have discussed in this section shows there is little room for algebraic manouvers (Wesson and Liu 1997). In canonical coordinates, the rest mass of an uncharged particle is l, the mass of a charged particle is $l\left(1+n^2/\Phi^2\right)^{1/2}$, and the charge itself is nl. The charge/mass ratio which appears in the Lorentz law is $q/m_q = n\left(1+n^2/\Phi^2\right)^{-1/2}$. All of these 4D physical quantities can be derived from 5D geometrical ones.

7.6 Particle Physics and Geometry

The fact that we can derive the 4D rest mass and electric charge of a particle from 5D geometry, as shown in the preceding section, raises the question of whether all properties of elementary particles are derivable in this way. Conceptually, this is an enormously attractive prospect. Technically, it appears at first sight to be difficult. Perhaps the first objection is that particle properties such as momenta are commonly regarded as localized, whereas geometry is commonly regarded as extended. However, this view is wrong from both directions. In wave-mechanics and quantum field theory, a particle is spread through (3D) space, and only becomes localized when we observe it. In general relativity and similar theories, the defining field

equations are local, telling us nothing about global properties such as topology until we obtain a solution with boundary conditions. In principle, there is no reason why local particle properties such as momenta should not be derived from local field equations such as those we have proposed above ($R_{AB} = 0$). Another objection to putting particles into the geometrical scheme is that most particles are not affected much by gravity or electromagnetism, whereas geometrical theories of physics like those of Einstein and Kaluza-Klein are designed to handle exactly these forces. There is cogency to this argument. Reintroducing the fundamental constants we discussed in Chapter 1, we can consider the characteristic length scales associated with mass, charge and spin. These are Gm/c^2, $(Gq^2/c^4)^{1/2}$ and $(G\hbar/c^3)^{1/2}$, and for typical elementary particles have sizes of order 10^{-56}, 10^{-34} and 10^{-32} cm. In other words, most particles are spin-dominated. However, there are cracks in this objection. Firstly, the "dressed" quantities which figure in the preceding numerology are not necessarily the same as the "bare" ones which the underlying theory presumably deals with (see Wesson 1992 for comments on this and models of elementary particles based on general relativity). Secondly, the fact that spin can be made into a length at all implies that it is geometric in nature. Our view, based on what has been stated above, is that it is worthwhile to try and extend the geometrical approach to particle physics.

Let us consider a simple case, namely that of a spinless, neutral particle. Our objective is to calculate the mass of a particle by following a line of reasoning that joins field theory and quantum theory (Liu and Wesson 1998). Particle mass is a 4D quantity, so we expect it to be determined by the 4D subspace (which is generally curved) of a 5D manifold (which may be flat). Let the particle have 4-momenta p_α and mass m. In classical mechanics p_α is a 4-vector, while in quantum mechanics it is commonly treated by introducing the de Broglie wavelengths which in a symbolic fashion can be written h/p_α. We can imagine that the de Broglie

wavelengths are related to the dimensions of the principle curvatures of the 4D subspace, or alternatively to the Ricci tensor $R_{\alpha\beta}$ or the Einstein tensor $G_{\alpha\beta} \equiv R_{\alpha\beta} - Rg_{\alpha\beta}/2$. For these latter, G = - R is an obvious scalar to relate to the scalar mass m. This leads us to conjecture that the mass of a particle is related to the curvature of the space it inhabits by a relation of the form

$$m^2 = \frac{k h^2 R}{c^2} \quad . \tag{7.42}$$

Here k is a dimensionless coupling constant, to be determined by a detailed analysis to which we now proceed.

Consider a metric of the form

$$dS^2 = g_{AB} dx^A dx^B = \Phi^2 ds^2 - \Phi^{2b} dl^2$$

$$ds^2 \equiv g_{\alpha\beta} dx^\alpha dx^\beta \quad . \tag{7.43}$$

This is fairly general, but should not be confused with the canonical metric (see above). In fact, we will assume $\Phi = \Phi(x^\alpha)$ in order to make contact with the conventional wave function of quantum mechanics (see below). The Christoffel symbols for (7.43) are

$$\hat{\Gamma}^\alpha_{\beta\gamma} = \Gamma^\alpha_{\beta\gamma} + \Phi^{-1}\left(\delta^\alpha_\beta \Phi_\gamma + \delta^\alpha_\gamma \Phi_\beta - g_{\beta\gamma}\Phi^\alpha\right)$$

$$\hat{\Gamma}^\alpha_{\beta\alpha} = \Gamma^\alpha_{\beta\alpha} + 4\Phi^{-1}\Phi_\beta$$

$$\Gamma^\alpha_{44} = b\Phi^{2b-3}\Phi^\alpha$$

$$\Gamma^\alpha_{\beta 4} = \Gamma^4_{\alpha\beta} = \Gamma^4_{44} = 0$$

$$\Gamma^4_{4\alpha} = b\Phi^{-1}\Phi_\alpha$$

$$\Gamma^D_{\alpha D} = \Gamma^\delta_{\alpha\delta} + (b+4)\Phi^{-1}\Phi_\alpha$$

$$\Gamma^D_{4D} = 0 \tag{7.44}$$

Here as elsewhere $\Phi_\alpha \equiv \Phi_{,\alpha}$, $\Phi^\alpha = g^{\alpha\beta}\Phi_\beta$, a repeated index implies a sum, and hats denote the 4D parts of 5D quantities. The 5D Ricci tensor can be written in terms of the 4D Ricci tensor as usually defined plus other terms, thus:

$$\hat{R}_{\alpha\beta} = R_{\alpha\beta} - (b+2)\Phi^{-1}\Phi_{\alpha;\beta} - \left(b^2 - 3b - 4\right)\Phi^{-2}\Phi_\alpha\Phi_\beta - \Phi^{-2}\left[\Phi\Phi^\gamma{}_{;\gamma} + (b+1)\Phi^\gamma\Phi_\gamma\right]g_{\alpha\beta}$$

$$R_{4\alpha} = 0$$

$$R_{44} = b\Phi^{2b-4}\left[\Phi\Phi^\alpha{}_{;\alpha} + (b+1)\Phi^\alpha\Phi_\alpha\right]. \tag{7.45}$$

Clearly the electromagnetic components of the field equations $R_{AB} = 0$ are trivially satisfied, while the remaining equations read

$$R_{\alpha\beta} = (b+2)\Phi^{-1}\Phi_{\alpha;\beta} + \left(b^2 - 3b - 4\right)\Phi^{-2}\Phi_\alpha\Phi_\beta \tag{7.46}$$

$$\Phi\Phi^\alpha{}_{;\alpha} + (b+1)\Phi^\alpha\Phi_\alpha = 0\,. \tag{7.47}$$

These are a set of 10 Einstein-like equations and 1 wave-like equation, and are what we will be concerned with in what follows.

From (7.46) and (7.47) we find that the 4D Ricci scalar is

$$R = -6(b+1)\Phi^{-2}\Phi^\alpha\Phi_\alpha\,. \tag{7.48}$$

This combined with (7.46) allows us to form the Einstein tensor, and Einstein's equations $G_{\alpha\beta} = 8\pi T_{\alpha\beta}$ then define an induced energy-momentum tensor given by

$$8\pi T_{\alpha\beta} = (b+2)\Phi^{-1}\Phi_{\alpha;\beta} + (b+1)(b-4)\Phi^{-2}\Phi_\alpha\Phi_\beta + 3(b+1)\Phi^{-2}\Phi^\gamma\Phi_\gamma g_{\alpha\beta}\,. \tag{7.49}$$

This obviously contains a term that is second order in the derivatives of the scalar field, and two terms that are first order as in flat-space quantum field theory (Roman 1969; Mandl and Shaw 1993). Because of this, we choose b = -2 at this point. This is also the choice that is compatible with the 5D harmonic coordinate condition, which implies wave-like solutions (Liu and Wanzhong 1988). Then the 5D and 4D line elements are

$$dS^2 = \Phi^2 ds^2 - \Phi^{-4} dl^2$$

$$ds^2 \equiv g_{\alpha\beta} dx^\alpha dx^\beta \tag{7.50}$$

where $g_{\alpha\beta} = g_{\alpha\beta}(x^\gamma)$, $\Phi = \Phi(x^\gamma)$. And the 5D field equations $R_{AB} = 0$ yield the 4D ones

$$R_{\alpha\beta} = 6\Phi^{-2}\Phi_\alpha \Phi_\beta \tag{7.51}$$

$$\Phi\Phi^\alpha{}_{;\alpha} - \Phi^\alpha \Phi_\alpha = 0 \ , \tag{7.52}$$

with an effective 4D energy-momentum tensor

$$8\pi T_{\alpha\beta} = 6\Phi^{-2}\left(\Phi_\alpha \Phi_\beta - \Phi^\gamma \Phi_\gamma g_{\alpha\beta}/2\right) \ . \tag{7.53}$$

These relations are expected to define the geometrical properties of a scalar wave which in terms of quantum theory would be described as a massive particle.

In section 7.5, we reviewed the conventional way of describing the 4D dynamics of a particle in classical theory. For the situation here, the Lagrangian is $L = m(ds/d\lambda) = m\left[g_{\alpha\beta}\left(dx^\alpha/d\lambda\right)\left(dx^\beta/d\lambda\right)\right]^{1/2}$, where λ is a parameter for the path and the mass is assumed to be a constant. The momenta are $p_\alpha = \partial L/\partial\left(dx^\alpha/d\lambda\right) = m\, g_{\alpha\beta}\left(dx^\beta/ds\right)$, with $p^\alpha = g^{\alpha\beta}p_\beta = m\left(dx^\alpha/ds\right)$, so $p_\alpha p^\alpha = m^2$. The action corresponding to the Lagrangian is $I = \int L d\lambda = \int m ds = \int m\left[g_{\alpha\beta}\left(dx^\alpha/ds\right)\left(dx^\beta/ds\right)\right]ds = \int p_\alpha dx^\alpha$. However, in place of the action we can use the wave function of quantum theory:

$$\Psi \equiv e^{-iI/\hbar} \ . \tag{7.54}$$

In terms of this, the momenta are given by

$$i\hbar \frac{\partial}{\partial x^\alpha}\Psi = p_\alpha \Psi \ . \tag{7.55}$$

If the p_α were constants as in flat spacetime, they would be the eigenvalues and Ψ the eigenfunction of the 4D momentum operator $i\hbar\partial/\partial x^\alpha$. Now the p_α are not generally constants

in curved spacetime, but we can keep (7.55) as the defining relation for the momenta because the preceding relations are covariant. Then following the philosophy of induced-matter theory, the question is how to relate the quantum relations (7.54), (7.55) in Ψ to the classical relations (7.51), (7.52) in Φ.

We answer this by adopting the ansatz

$$\Phi = \Psi^{\varepsilon} , \tag{7.56}$$

where ε is a dimensionless arbitrary parameter. Then (7.51), (7.52) give

$$R_{\alpha\beta} = -\frac{6\varepsilon^2}{\hbar^2} p_\alpha p_\beta \tag{7.57}$$

$$\Psi^{\alpha}{}_{;\alpha} + \frac{m^2}{\hbar^2}\Psi = 0 . \tag{7.58}$$

The corresponding induced energy-momentum tensor (7.53) is

$$8\pi T_{\alpha\beta} = -\frac{6\varepsilon^2}{\hbar^2}\left(p_\alpha p_\beta - \frac{m^2 g_{\alpha\beta}}{2} \right) . \tag{7.59}$$

These relations satisfy the picture we set out to paint: (7.57) says that the de Broglie wavelengths associated with p_α describe by their product $p_\alpha p_\beta$ the curvature of the 4D space as represented by the Ricci tensor $R_{\alpha\beta}$; (7.58) says there is a Klein-Gordon equation for a particle of mass m which involves by (7.56) and (7.50) the scalar field, or alternatively a conformal factor applied to ordinary spacetime; (7.59) says there is an "energy-momentum" tensor which is actually a dynamical object and defines via the Bianchi identities $T^{\alpha}_{\beta;\alpha} = 0$ the motion of the particle. The last point may stand some elaboration (Liu and Wesson 1998). Thus we note that $\left(p^\alpha p^\beta - m^2 g^{\alpha\beta}/2\right)_{;\beta} = 0$ gives $p^{\beta}{}_{;\beta} = 0$, which substituted back gives $p^{\alpha}{}_{;\beta}\, p^{\beta} = 0$, the standard 4D equations of motion; and it can be shown by some tedious algebra that the 5D geodesic (7.10) for metric (7.50) leads to the same equations. Lastly, we see that the trace of (7.57) yields

$R=-6\varepsilon^2 m^2/\hbar^2$, which agrees with our starting relation (7.42) and implies that the mass of a particle is related to the scalar curvature of the 4D space it inhabits.

To go further, we need exact solutions. The most convenient way to show such is to write the 4D part of the metric as

$$ds^2 = dt^2 - e^{2\lambda}dx^2 - e^{2\mu}dy^2 - dz^2 \ , \tag{7.60}$$

where $\lambda=\lambda(t,z)$ and $\mu=\mu(t,z)$. This metric can describe a plane wave moving in the z-direction (Kramer et al. 1980; Liu and Wanzhong 1988). We expect the associated particle to have momenta

$$p_0 = m\frac{dt}{ds} \equiv E, \qquad p_1=0, \qquad p_2=0, \qquad p_3=-m\frac{dz}{ds}\equiv -p \tag{7.61}$$

These particle properties are connected to the wave properties by the field equations (7.57), (7.58). The components of (7.57), using (7.60) and (7.61), read

$$\begin{aligned}
R_{00} &= -(\lambda+\mu)_{,00} - \left(\lambda_{,0}^2+\mu_{,0}^2\right) = -6\varepsilon^2\hbar^{-2}E^2 \\
R_{03} &= -(\lambda+\mu)_{,03} - \left(\lambda_{,0}\lambda_{,3}+\mu_{,0}\mu_{,3}\right) = 6\varepsilon^2\hbar^{-2}Ep \\
R_{33} &= -(\lambda+\mu)_{,33} - \left(\lambda_{,3}^2+\mu_{,3}^2\right) = -6\varepsilon^2\hbar^{-2}p^2 \\
R_{11} &= e^{2\lambda}\left[\lambda_{,00}-\lambda_{,33}+\lambda_{,0}(\lambda+\mu)_{,0}-\lambda_{,3}(\lambda+\mu)_{,3}\right]=0 \\
R_{22} &= e^{2\mu}\left[\mu_{,00}-\mu_{,33}+\mu_{,0}(\lambda+\mu)_{,0}-\mu_{,3}(\lambda+\mu)_{,3}\right]=0 \\
R_{\alpha\beta} &= 0 \quad \text{(otherwise)} \ .
\end{aligned} \tag{7.62}$$

It is obvious when written in this way that there is a solution given by

$$\lambda=-\mu=\sqrt{3}\varepsilon\hbar^{-1}(Et-pz) \ . \tag{7.63}$$

This with (7.54) gives the action and wave function as

$$I=\int p_\alpha dx^\alpha = Et-pz \tag{7.64}$$

$$\Psi = e^{-i(Et-pz)/\hbar} \quad . \tag{7.65}$$

The last should satisfy the fifth or Klein-Gordon part of the field equations (7.58), and we find that it does so. The induced or effective 4D energy-momentum tensor by (7.59) and (7.61) has the following nonzero components:

$$\begin{aligned} 8\pi T_0^0 &= -6\varepsilon^2\hbar^{-2}\left(E^2 - m^2/2\right) \\ 8\pi T_3^0 &= -T_0^3 = 6\varepsilon^2\hbar^{-2}Ep \\ 8\pi T_1^1 &= T_2^2 = 3\varepsilon^2\hbar^{-2}m^2 \\ 8\pi T_3^3 &= 6\varepsilon^2\hbar^{-2}\left(p^2 + m^2/2\right) . \end{aligned} \tag{7.66}$$

These obey $8\pi T_\alpha^\alpha = 6\varepsilon^2\hbar^{-2}m^2$, and we recall that E, p, m are related by $p_\alpha p^\alpha = E^2 - p^2 = m^2$. This constraint means that we have a class of exact solutions that depends on 2 independent dynamical constants, E and p. [The dimensionless constant ε which we introduced in (7.56) to connect the scalar field potential and the wave function could in principle be absorbed into E and p, or into $\hbar$.] We can sum up the 5D wave solution as follows:

$$dS^2 = \Psi^{2\varepsilon}ds^2 - \Psi^{-4\varepsilon}dl^2 \tag{7.67}$$

$$ds^2 = dt^2 - e^{2\lambda}dx^2 - e^{-2\lambda}dy^2 - dz^2 \tag{7.68}$$

$$\Psi = e^{-i(Et-pz)/\hbar}, \quad \lambda = \sqrt{3}\varepsilon\hbar^{-1}(Et - pz) . \tag{7.69}$$

We have examined algebraic properties of this class of solutions using a fast software package. Apart from confirming $R_{AB} = 0$, we find $R_{ABCD} \neq 0$ so these solutions are not 4D curved and 5D flat like those considered in Section 4.7 (for example, for the above solutions we find that $R_{1414} = -2\varepsilon^2\left(3E^2 + p^2\right)\Psi^{-4\varepsilon} \neq 0$ and $K \equiv R_{ABCD}R^{ABCD} \neq 0$). However, while these solutions are in general curved, they have $\sqrt{-g} = 1$ for the determinant of the 4D metric and so are special algebraically.

This last point, plus other choices we have made along the way, leave broad scope for a more general examination of the idea that particle rest mass is related to spacetime curvature. We finish this section by focussing on two issues that need to be examined before such a Machian view can be adopted. First, it is not always possible to write the Ricci tensor as the product of two (momentum) vectors, as in (7.57). More work is needed on this presumably using the Segre classification of the Ricci tensor (Paiva, Reboucas and Teixeira 1997). Second, it is possible to change the conformal factor on the 4D spacetime part of a 5D metric (7.43), which changes the mass as defined by (7.42). For example, consider two 4D metrics related by a conformal transformation, $g_{\alpha\beta} = e^{\Omega(x)} \overline{g}_{\alpha\beta}$, with Ricci scalars R and $\overline{R}$ (Synge 1960). Then

$$R = e^{-\Omega}\left[\overline{R} - 3\overline{\Box}\Omega + (3/2)\overline{g}^{\alpha\beta}\Omega_{\alpha}\Omega_{\beta}\right] \quad . \tag{7.70}$$

This means in the induced-matter picture that a massless particle $\left(\overline{R}=0\right)$ becomes massive $(R \neq 0)$ by virtue of a conformal transformation of the 4D metric. In other words, conformal transformations in classical field theory are the analog of the Higgs mechanism in quantum theory.

7.7 Conclusion

The canonical coordinate system for 5D metrics leads to major physical insights. The 1-body solution (7.1) involves no departure from conventional dynamics, so from the classical tests in the solar system there is no way to tell if we should use the 5D metric or only its 4D part. The inflationary cosmological solution (7.7) leads to the same conclusion. But in both cases, the metric only exists if there is a finite cosmological constant, or equivalently a finite energy density for the vacuum; so in principle a way to differentiate between 5D and 4D is to use thermodynamics. Another way is to use a spinning object, since the equations (7.10), (7.11)

which govern this effectively lift degeneracy, and result in a precession formula (7.20) which picks up a term from the fifth dimension.

In the canonical frame, a comparison of 4D and 5D dynamics shows that the momenta of particles are simply related by the ratio of the intervals, and that the charge/mass ratio of a particle (7.35) follows from this and is geometrical in nature. A closer examination shows that the mass or energy of an uncharged particle (7.40) is just $x^4 = l$, and that the electric charge (7.41) is just a multiple of this. In other words, in the canonical frame, mass is a coordinate and charge is a momentum.

However, all coordinate frames are valid in 5D relativity, so the question arises of how to define the mass of a particle in a general, non-canonical, frame. Following the induced-matter philosophy, it is reasonable to suppose that the momenta of a particle, or its de Broglie wavelengths, are related to the principal curvatures of the 4D subspace it inhabits; and that its mass is related to the scalar curvature of that subspace. The latter condition is easier to quantify as in (7.42), which says that the mass (squared) is proportional to the Ricci scalar. The former condition is harder to quantify, but as in (7.57) implies a relation between the momenta (producted) and the Ricci tensor. We obviously need more work on this, especially on how the conformal factor for the 4D space affects the Ricci scalar (7.70) and thereby the mass. However, the results of Section 7.6 for particles run parallel to those of Section 4.4, which showed the existence of classical vacuum waves with complex metric coefficients. Then a reasonable inference is that the wave function of quantum theory is related to a complex conformal function applied to classical spacetime.

References (Chapter 7)

Barbour, J.B., Pfister, H. (eds.) 1995, Mach's Principle: From Newton's Bucket to Quantum Gravity (Birkhauser, Boston).

Hoyle, F., Narlikar, J.V. 1974, Action at a Distance in Physics and Cosmology (Freeman, San Francisco).

Jammer, M. 1961, Concepts of Mass in Classical and Modern Physics (Harvard U. Press, Cambridge).

Jantzen, R.T., Keiser, G.M., Ruffini R. (eds.) 1996, Proc. 7th Marcel Grossman Meeting (World Scientific, Singapore).

Kalligas, D., Wesson, P.S., Everitt, C.W.F. 1995, Astrophys. J. 439, 548.

Kramer, D., Stephani, H., Herlt, E., MacCallum, M., Schmutzer, E. 1980, Exact Solutions of Einstein's Field Equations (Cambridge U. Press, Cambridge).

Liu, H., Wanzhong, S. 1988, Gen. Rel. Grav. 20, 407.

Liu, H., Wesson, P.S. 1996, Class. Quant. Grav. 13, 2311.

Liu, H., Wesson, P.S. 1998, Int. J. Mod. Phys. D 5, 737.

Mandl, F., Shaw, G. 1993, Quantum Field Theory (Wiley, New York), 29.

Paiva, F.M., Reboucas, M.J., Teixeira, A.F.F. 1997, J. Math. Phys. 38, 4228.

Roman, P. 1969, Introduction to Quantum Field Theory (Wiley, New York), 19.

Synge, J.L. 1960, Relativity: The General Theory (North Holland, Amsterdam), 318.

Wesson, P.S. 1992, Space Sci. Rev. 59, 365.

Wesson, P.S., Liu, H. 1997, Int. J. Theor. Phys. 36, 1865.

Wesson, P.S., Mashhoon, B., Liu, H. 1997, Mod. Phys. Lett A 12, 2309.

8. RETROSPECT AND PROSPECT

"To put the conclusion crudely – the stuff of the world is mind-stuff"

(Sir A.S. Eddington, Cambridge, 1939)

By this quote, Eddington did not mean that we create the world inside our heads. He rather meant that we perceive an external world, but strongly influenced by the physiological and psychological constraints that attach to us being human. This view was one he was led to after experiencing and partaking in the development of relativity and quantum theory in the first half of the twentieth century. More recent developments, including the ones outlined in this book, bolster the view that physics is not discovered so much as invented.

In Chapter 1, we saw that fundamental constants can be brought into and removed from physics as a matter of convenience. In Chapter 2, Riemannian geometry in 5D was used to remove the artificial distinction between the field and its source: what we call matter is, at least at the classical level, the result of geometry. This induced-matter theory uses the field equations $R_{AB}(x^C) = 0$, and is algebraically much richer than old Kaluza-Klein theory with $G_{AB}(x^\alpha) = 8\pi T_{AB}(x^\alpha)$ or Einstein's theory with $G_{\alpha\beta}(x^\gamma) = 8\pi T_{\alpha\beta}(x^\gamma)$. But it contains general relativity, and as shown in Chapter 3 the 1-body soliton solutions agree with the classical tests. In Chapter 4, we used exact solutions of $R_{AB} = 0$ with their associated $T_{\alpha\beta}(x^A)$ to examine a number of systems of cosmological and astrophysical importance. The standard 4D cosmology, consisting of the radiation-dominated solution for early times and the dust (Einstein-de Sitter) solution for late times, is curved in 4D but flat in 5D. Mathematically, we have to conclude that the 4D big bang is due to an unfortunate choice of 5D coordinates. Conceptually, it is subjective or invented in the Eddingtonian sense. Electromagnetism was treated in Chapter 5 as in the

original theory of Kaluza, and charged solitons and black holes if they should be detected astrophysically would differentiate between 4D and 5D. In Chapter 6, the idea of a canonical metric was introduced. While natural, this has radical implications: the world model does not exist unless there is a fifth dimension, and if the 4D part of the model depends on the extra coordinate there exists a fifth force. The ramifications of the canonical metric were examined in Chapter 7. There is a 1-body class of solutions which "intersects" the soliton solutions insofar as both classes contain the unique 4D Schwarzschild solution. For such solutions, we cannot tell if the world is 5D or 4D merely by looking at conventional dynamics, but other effects including spin can show the difference. The canonical scheme can be extended to particle physics, and perhaps its most remarkable implication concerns the rest mass of a particle: there is at least one coordinate system where the extra coordinate plays the role of mass.

However, the unification of space, time and matter we have outlined in the preceding chapters leads to some questions:

(a) Is there anything special about 5D Riemannian space? The answer is almost certainly, NO. To a first approximation, the world locally is 1D in nature $\left(x^0 >> x^{123}\right)$. To a better approximation it is described by 4D spacetime. Classical matter in spacetime can be accommodated by a 5D manifold. But while ND Riemannian spaces have special algebraic properties for small N, there is no reason to believe that there is some unique or chosen value for it. We choose N in accordance with what physics we wish to explain. (This argument also applies to N = 10, 11 as in superstrings and supergravity.) Indeed, we might wish to consider $N \to \infty$. A related possibility is that we may wish to go beyond symmetric Riemannian space to consider non-symmetric spaces or ones with torsion, or more general spaces like that of Finsler, all of which offer more algebraic opportunities that might be utilized for physics.

(b) The problem of zero-point fields in particle physics may be solved by supersymmetry; and the equivalent (cosmological constant) problem in general relativity may be solved by a scalar field. But more work is needed on the connection between these approaches. In a classical 5D field theory like the one developed above, canonical metrics necessarily have a parameter that plays the role of a 4D cosmological constant, so there is equivalently a vacuum energy. However, if we adhere to the philosophy of Einstein, such properties of the vacuum are the result of coordinate choices, and should not be elevated to any higher status (as done, for example, in the theory of quantum electrodynamic vacuum polarization). We do not know how many scalar fields, or equivalently what number of dimensions, are necessary to resolve these problems with energy.

(c) The canonical 1-body solution is quite different from the soliton solution, and we do not know which describes the field outside a massive object such as the Sun. We suspect that it is the former, which implies some other physical identification for the solitons.

(d) In the canonical 5D metric, the extra coordinate plays the role of particle rest mass. But the former can be negative, so can mass be negative also? The situation here is in some ways analogous to the one encountered by Dirac when he was considering the theory of the electron, and of course he was led to the proposal of the positron. In gravitational theory, Bonnor has shown that there is nothing intrinsically contradictory about negative mass. However, in a world where positive and negative mass both exist, the detection of an object with negative mass would be problematical because it would repel other particles. Thus while 5D theory naively predicts particles with negative mass, it is not clear how such particles could be detected or why the world appears to consist predominantly of objects with positive mass.

(e) For canonical metrics whose 4D part depends on the extra coordinate, there is a fifth force. Why has this hitherto escaped detection? A kind of answer to this can be given by appeal to the weak equivalence principle, which implies that the 4D acceleration of a test particle is independent of its mass. However, this really makes the equivalence principle a symmetry of the metric, and does not yield a deeper explanation. A related question is why particle masses appear (at least locally) to be constants, whereas the theory allows them to be variable. A technical answer is that the coordinate system is one where the fifth component of the 5-velocity is zero (i.e., x^4 is comoving). However, this does not explain why 4D physics has evolved to use masses as units.

(f) If a fifth force exists, it may only be apparent in microscopic systems, as opposed to the macroscopic ones we have mainly considered in the preceding account. We can ponder what the effects would be of 5D physics applied to elementary particles if we tried to interpret the latter in 4D. (A different approach to this would be to reformulate quantum mechanics in 5D, which however we leave as an exercise for the reader.) To be specific: what would be the effects of exact 5D conservation laws if we only considered their 4D parts? A possibility is that effects commonly attributed to Heisenberg's uncertainty principle in 4D could instead be understood as the consequences of deterministic laws in 5D.

This last idea, while speculative, would have appealed to Einstein. He had the good fortune (and skill) to succeed at most of the physics he took up. A lesson to be learned from his work is that a powerful idea should be pushed forward; and it appears to us that the principle of covariance is one of the most powerful ideas in modern physics. Field equations involving tensors have enormous algebraic scope if we do not restrict the group of coordinate invariance and do not restrict the number of dimensions. In 5D, we can realize Einstein's vision of

transforming the "base-wood" of matter into the "marble" of geometry. The trick is simply to keep the theory general algebraically and to make the appropriate identifications physically. While we do not think it is the only way, we are led to the opinion that the world can be described if we so choose in terms of pure geometry.

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